International Journal of Modern Physics A, Vol. 10, No. 17 (1995) 2537–2577

REDUCTION OF TODA LATTICE HIERARCHY TO GENERALIZED KdV HIERARCHIES AND THE TWO-MATRIX MODEL

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Received 20 October 1994

Toda lattice hierarchy and the associated matrix formulation of the $2M$-boson KP hierarchies provide a framework for the Drinfeld–Sokolov reduction scheme realized through Hamiltonian action within the second KP Poisson bracket. By working with free currents, which Abelianize the second KP Hamiltonian structure, we are able to obtain a unified formalism for the reduced $\mathrm{SL}(M+1,\ M-k)$ KdV hierarchies interpolating between the ordinary KP and KdV hierarchies. The corresponding Lax operators are given as superdeterminants of graded $\mathrm{SL}(M+1,\ M-k)$ matrices in the diagonal gauge and we describe their bracket structure and field content. In particular, we provide explicit free field representations of the associated $W(M, M-k)$ Poisson bracket algebras generalizing the familiar nonlinear W_{M+1} algebra. Discrete Bäcklund transformations for $\mathrm{SL}(M+1,\ M-k)$ KdV are generated naturally from lattice translations in the underlying Toda-like hierarchy. As an application we demonstrate the equivalence of the two-matrix string model to the $\mathrm{SL}(M+1,1)$ KdV hierarchy.

1. Introduction

Integrable Hamiltonian systems occupy an important place in diverse branches of theoretical physics as exactly solvable models of fundamental physical phenomena ranging from nonlinear hydrodynamics to string theory of elementary particle interactions at ultra high energies,[1–3] including high energy QCD in $3+1$ space–time

*Work supported in part by the US Department of Energy under contract DE-FG02-84ER40173.
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‡Work supported in part by CNPq.

dimensions.[4] Among the most notable physically relevant integrable models is the Kadomtsev–Petviashvili (KP) hierarchy of integrable soliton nonlinear evolution equations.[5,6] The main interest in the KP hierarchy during the last few years stems from their deep connection with string (multi)matrix models.[7]

The general KP system is a $(1+1)$-dimensional integrable model containing an infinite number of fields. Its reductions with a finite number of fields ("multiboson KP hierarchies" for short) are, similarly, integrable systems which naturally appear in two different but related settings. As continuous (field-theoretic) integrable models they define a consistent *Poisson reduction* of the complete continuous KP system.[8] However, they can also be traced back to the *discrete* Toda lattice hierarchy.[9–12] These two distinct formulations are in agreement due to the existence of a discrete symmetry for the continuous multiboson KP hierarchy, which is a canonical mapping realized by a similarity transformation for the underlying Lax operators.[13,14] The presence of this discrete transformation allows us to view the Toda lattice[15] as a union of sites, each being a gauge copy of one continuous multiboson KP system. Throughout this paper we shall benefit essentially from this relation between the Toda lattice and the continuous multiboson KP system.

It has long been known that the most simple example of the multiboson KP systems, the two-boson KP system, contains the usual KdV hierarchy.[16] In Ref. 17 the Dirac reduction of the two-boson KP system was proposed and the KdV hierarchy was obtained in this process. Subsequently, the Dirac reduction was applied to other multiboson KP systems, which resulted in a large family of generalized KP–KdV hierarchies.[12,18]

In this paper we investigate the reduction of the multiboson KP hierarchies employing the Drinfeld–Sokolov (DS) reduction scheme realized in a nonconventional way — as Hamiltonian reduction within the second KP Poisson bracket structure. By working with free currents Abelianizing the latter highly reducible Poisson structure, we are able to obtain a unified description of the various reduced hierarchies, their bracket structure and their field content. Let us recall that the KP hierarchy is endowed with bi-Hamiltonian Poisson bracket structures resulting from the two compatible Hamiltonian structures on the algebra of pseudodifferential operators.[9] The latter are given by

$$\{\langle L|X\rangle, \langle L|Y\rangle\}_1 = -\langle L|[X,Y]\rangle\,, \tag{1.1}$$

$$\{\langle L|X\rangle, \langle L|Y\rangle\}_2 = \mathrm{Tr}_A[(LX)_+LY - (XL)_+YL]\,. \tag{1.2}$$

Here and below the following notations are used. $\langle\cdot|\cdot\rangle$ denotes the standard bilinear pairing via the Adler trace $\langle L|X\rangle = \mathrm{Tr}_A(LX)$ with $\mathrm{Tr}_A X = \int \mathrm{Res}\, X$. Here L, X, Y are arbitrary elements of the algebra of pseudodifferential operators of the form $L = \sum_{k\geq-\infty} u_k D^k$, $X = \sum_{k\geq-\infty} D^k X_k$, where $D = \partial/\partial x$ denotes the differential operator w.r.t. x. Furthermore, the subscripts $\pm$ in $X_\pm$ denote taking the purely differential or the purely pseudodifferential part of X, respectively. The Lax operator of the KP hierarchy has the specific form $L = D + \sum_{k=1}^{\infty} u_k D^{-k}$, and therefore the second Poisson bracket (1.2) is modified to

$$\{\langle L|X\rangle, \langle L|Y\rangle\}_2 = \mathrm{Tr}_A[(LX)_+LY-(XL)_+YL]+\int dx\, \mathrm{Res}([L,X])\partial^{-1}\, \mathrm{Res}([L,Y])\,. \tag{1.3}$$

The last term is a Dirac bracket term originating from the second class constraint $u_0 = 0$.

The Toda lattice hierarchy in the matrix form is a central object in our study of Hamiltonian reduction. Up to a phase gauge transformation, the Toda matrix of the associated linear problem has the form of a matrix in the DS gauge with an extra traceless diagonal part. Hence, there exists a residual gauge symmetry preserving the DS-like form of the Toda matrix. Correspondingly, the space of Toda matrices splits into orbits of this residual gauge group. The reduction is then accomplished in the final step by restricting to the symplectic quotient space (by quotienting out the residual gauge symmetry). In the case under consideration the final step involves removal of the diagonal terms (currents) of the Toda hierarchy matrix. One obtains in this process various generalized KP–KdV hierarchies with the usual KdV model corresponding to the case where all the diagonal currents are gauged away.

Such a reduction, when based on matrix calculations, becomes quickly cumbersome with the increasing rank of the matrices. The question is whether there exists a convenient way of handling the residual gauge transformations. The natural symplectic Kirillov–Kostant–Symes (KKS) form associated with the space of Toda matrices degenerates on the vector fields tangent to the orbits defined by the residual gauge transformations. Hence, one does not expect the action of the residual gauge transformation to be Hamiltonian as there is no natural KKS-type Poisson bracket associated with the linear Toda matrix problem before the last step of Hamiltonian reduction is taken. Surprisingly, it turns out that the relevant gauge group action is nevertheless Hamiltonian but with respect to the second Poisson bracket (1.3) of the multiboson KP system. This enables us to describe the DS reduction within the framework of the Poisson manifolds and to find closed expressions for the gauge-transformed quantities on the reduced manifold.

The Abelianized representation, i.e. the representation in terms of free current, of the second Poisson bracket[20] plays a key role in the above formalism. Here the underlying lattice structure is the Volterra lattice and the problem is transformed from the DS gauge of the Toda hierarchy to the diagonal gauge. In this representation the residual gauge transformations take a simple form and the constraint manifold is described directly in terms of the original Abelian canonical variables. It is crucial that the second bracket structure is reducible and the residual gauge symmetry of the DS problem triggers a total factorization of the bracket structure.

The corresponding Lax operators appearing on various levels of reduction are constructed in terms of currents spanning the Cartan subalgebra of the graded $\mathrm{SL}(M+1,\ M-k)$ Kac–Moody algebra.[21] The variable k labels the level of reduction with $k = 0$ corresponding to the original $2M$-boson KP system and $k = M$ describing the maximal reduction to the ordinary $\mathrm{SL}(M+1)$ KdV hierarchy. In the latter case the Lax operator reduces to the simple determinant of the Fateev–Lukyanov type.[22] Hence, we obtain a unified formalism for the reduced $\mathrm{SL}(M+1,\ M-k)$ KdV hierarchies (KP–KdV hierarchies), which interpolate between the original KP systems and the ordinary KdV hierarchies. The generic Lax operators are given as superdeterminants of the graded $\mathrm{SL}(M+1,\ M-k)$ matrices in the diagonal gauge. We describe their bracket structure. Thanks to our Abelianization technique we are also able to give a free field construction for all dynamical variables of the $\mathrm{SL}(M+1,\ M-k)$ KdV hierarchies.

The generalized $\mathrm{SL}(M+1,1)$ KdV hierarchies have recently been encountered in the study of the two-matrix model.[23,24] In Ref. 24 the simplest nontrivial Toda-like lattice integrable system, derived from the partition function of the two-matrix model with matrix potentials of orders $p_1 =$ arbitrary and $p_2 = 3$, was shown to be equivalent to the $(1+1)$-dimensional generalized $\mathrm{SL}(3,1)$ KdV hierarchy. In this paper we extend the above analysis to the case of arbitrary finite p_2.

The organization of the material is as follows. In Sec. 2 we recapitulate the basic facts about the Toda lattice hierarchy and the matrix approach to the spectral problem versus the continuum multiboson KP hierarchy. In Sec. 3 we compare Dirac and DS reductions of the two-boson KP hierarchy and present DS reduction for the four-boson hierarchy. Next, in Sec. 4, we show that the residual gauge transformation has a Hamiltonian action with respect to the second KP Poisson bracket and discuss how the DS reduction, described in the previous section, is induced in this Hamiltonian manner. These results are generalized to an arbitrary multiboson KP hierarchy in Sec. 5, where use is made of a set of free currents Abelianizing the second Poisson bracket structure. These currents enter into the Lax operator in a form which naturally leads to the graded $\mathrm{SL}(M+1,M)$ Kac–Moody algebra. In Sec. 6 the reduction process is shown to be equivalent to reducing the graded $\mathrm{SL}(M+1,M)$ algebra to $\mathrm{SL}(M+1,\ M-k)$ algebra. This framework allows for a simple expression for the second bracket structure of the $\mathrm{SL}(M+1,\ M-k)$ KdV hierarchy in terms of Lax operators being superdeterminants of the graded $\mathrm{SL}(M+1,\ M-k)$ matrices in the diagonal gauge. Also, we show that the reduced generalized KP–KdV hierarchies are integrable (bi-Hamiltonian) and possess canonical discrete symmetries. Our construction provides explicit free field representations of the associated $W(M, M-k)$ Poisson bracket algebras generalizing the well-known nonlinear W_{M+1} algebra.[25] Discrete Bäcklund transformations for $\mathrm{SL}(M+1,\ M-k)$ KdV are generated naturally from lattice translations in the underlying Toda-like hierarchy. Finally, as an application we demonstrate in Sec. 7 the equivalence of the two-matrix string model to the $\mathrm{SL}(p_1,1)$ KdV hierarchy, where $p_{1,2}$ $(p_1 \leq p_2)$ are the orders of the matrix model potentials.

2. Toda Hierarchy Versus Multiboson KP Hierarchy

2.1. *Toda hierarchy and matrix approach to the spectral problem*

We start with the spectral equation,

$$\partial\Psi_n = \Psi_{n+1} + a_0(n)\Psi_n \,,$$
$$\lambda\Psi_n = \Psi_{n+1} + a_0(n)\Psi_n + \sum_{k=1}^{M} a_k(n)\Psi_{n-k} \,, \tag{2.1}$$

associated with the Toda lattice hierarchy (for the most general case, see Ref. 26). Here $\partial \equiv \partial_x \equiv \partial/\partial t_{1,1}$, where $t_{1,1}$ denotes the first lattice evolution parameter, which is now considered as a space coordinate of a $(1+1)$-dimensional integrable system. The spectral equation (2.1) can be rewritten as a matrix equation, $(\mathbb{1}\partial - \mathbf{Q})$ $\Psi = 0$, which in components is given by

$$\begin{pmatrix} \partial - a_0(n-M) & -1 & 0 & \cdots & 0 \\ 0 & \partial - a_0(n-M+1) & -1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \partial - a_0(n-1) & -1 \\ a_M(n) & a_{M-1}(n) & \cdots & a_1(n) & \partial - \lambda \end{pmatrix}$$
$$\times \begin{pmatrix} \Psi_{n-M} \\ \Psi_{n-M+1} \\ \vdots \\ \Psi_{n-1} \\ \Psi_n \end{pmatrix} = 0 \,. \tag{2.2}$$

Note first that by eliminating all Ψ_{n-i}, $i \neq 0$, from the set of equations represented by (2.2) we obtain

$$\lambda\Psi_n = L_n^{(M)}\Psi_n \,, \tag{2.3}$$

where

$$L_n^{(M)} = \partial + \sum_{k=1}^{M} a_k(n) \frac{1}{\partial - a_0(n-k)} \cdots \frac{1}{\partial - a_0(n-1)} \,. \tag{2.4}$$

The matrix form (2.2) of the Toda spectral problem will be a starting point of our discussion of DS reduction. We first perform a phase gauge transformation:

$$\Psi_{n-k} \to \exp\left[-\frac{1}{M+1}\sum_{i=1}^{M}\int a_0(n-i)\right]\Psi_{n-k} \,, \qquad 0 \le k \le M \,, \tag{2.5}$$

which transforms the diagonal terms in (2.2) to

$$\partial - a_0(n-i) \to \partial + \frac{1}{M+1}\sum_{j=1,\, j\neq i}^{M} a_0(n-j) - \frac{M}{M+1} a_0(n-i) \,,$$
$$\partial - \lambda \to \partial - \lambda + \frac{1}{M+1}\sum_{j=1}^{M} a_0(n-j) \,. \tag{2.6}$$

This transformation renders the matrix $\mathbf{Q}$ traceless (for $\lambda = 0$). Let us denote by

$$\mathcal{M}\Psi \equiv (\partial - \bar{\mathbf{Q}})\Psi = 0\,, \qquad \bar{\mathbf{Q}} = \mathcal{E} - \omega\,, \qquad \mathcal{E}_{ij} = \delta_{i+1,j}\,, \qquad \mathrm{tr}\,\omega = 0 \tag{2.7}$$

the equation obtained from (2.2) by the above phase transformation accompanied by setting $\lambda = 0$. Now, one can follow the ideas of the DS Hamiltonian reduction scheme.[27,28] Indeed, the space of Toda matrices $\mathcal{O}_{\mathrm{Toda}} = \{\bar{\mathbf{Q}}; \bar{\mathbf{Q}} \text{ as in } (2.7)\}$ can be viewed as a submanifold of the phase space of the $\mathrm{SL}(M+1)$ WZNW model, which is a coadjoint orbit of $\mathrm{SL}(M+1)$:

$$\mathcal{O}_G = \{S(g) = \partial g g^{-1};\ g \in G = \mathrm{SL}(M+1)\}\,. \tag{2.8}$$

Thus, this submanifold $\mathcal{O}_{\mathrm{Toda}}$ corresponds to a *partial* gauge-fixing of the first class constraints $\Phi \equiv \partial g g^{-1}|_{\mathcal{G}_+} - \mathcal{E} = 0$, $g \in \mathrm{SL}(M+1)$, in a gauged $\mathrm{SL}(M+1)$ WZNW model whose "big" phase space is (2.8). Here, as usual, $\mathcal{G}_\pm \subset \mathcal{G} = \mathrm{sl}(M+1)$ denote the nilpotent upper/lower-triangular subalgebras. Therefore, there exists a residual M-dimensional gauge symmetry group $\Gamma \subset \mathrm{SL}(M+1)$, $h \in \Gamma$:

$$\mathcal{M} \equiv \partial - \mathcal{E} + \omega \to h^{-1}\mathcal{M}h = \partial - \mathcal{E} + \bar{\omega}\,, \tag{2.9}$$

which preserves the form of the ω matrix and defines a gauge orbit for the Toda lattice hierarchy in the matrix form. The natural symplectic form (KKS form) on the "big" phase space (2.8) degenerates on the vector fields tangent to the gauge orbits (2.9). Hence, there is no natural KKS-type Poisson bracket structure for the coefficients $a_0(n-l)$, $a_l(n)$, $l = 1, \ldots, M$, of $\bar{\mathbf{Q}}$ associated with the linear matrix problem (2.7). Consequently, *a priori* one does not expect the action of the residual gauge transformations (2.9) to be Hamiltonian. We shall find later on that this gauge action can nevertheless be realized as a Hamiltonian action; namely, it is generated by the second KP Poisson bracket (1.3) for the $2M$-boson KP Lax operator (2.4) inherent in the Toda matrix spectral problem (2.2).

The determinant of $\mathcal{M}$ in (2.7) can be written as

$$\det\mathcal{M} = \partial^{M+1} + u_{M-1}\partial^{M-1} + \cdots + u_1\partial + u_0 \tag{2.10}$$

and clearly is invariant under (2.9). The differential operator $\det\mathcal{M}$ can also be obtained from $\mathcal{M}\Psi = 0$ [see (2.7)] by eliminating all $\Psi_{n-M+1}, \ldots, \Psi_n$ apart from Ψ_{n-M}. Generally we obtain a family of Lax operators in the process of eliminating all Ψ's apart from one element of the column in (2.2), which we denote by Ψ_{n-i_0}. The case $i_0 = M$ is the "pure" KdV Lax operator from (2.10) while $i_0 = 0$ gives the KP Lax operator $L_n^{(M)}$ from (2.4) (up to a phase gauge transformation). For $0 < i_0 < M$ we get a family of Lax operators invariant under various subgroups of the residual gauge symmetry (2.9). We shall implement in this paper the DS reduction scheme in the above-mentioned nonconventional setting — as a Hamiltonian reduction with respect to the second KP Poisson bracket — to describe this family of Lax operators contained in the linear system of (2.2).

2.2. *KP hierarchy: the first bracket*

From (2.1) we obtain the consistency conditions

$$\partial a_0(n) = a_1(n+1) - a_1(n)\,, \tag{2.11}$$

$$a_k(n) = a_k(n-1) + [\partial + a_0(n-k) - a_0(n-1)]a_{k-1}(n-1)\,, \quad k = 1, \dots, M\,, \tag{2.12}$$

$$\partial a_M(n) = a_M(n)[a_0(n) - a_0(n-M)]\,, \tag{2.13}$$

which are the Toda equations of motion. It is easy to see that all $a_k(n)$, $k = 0, 1, \dots, M$, at each lattice site n are expressed as functionals of only $2M$ independent functions, which can be chosen to be, for example, $a_0(M-k)$ and $a_k(M)$, $k = 1, \dots, M$.

We shall now relate the Toda lattice equation (2.12) to the recurrence relations for the $2M$-boson KP Lax operators derived in Ref. 8 and used there to Abelianize the first Poisson bracket structure (1.1). Let us introduce the correspondence [with $n = M$ in (2.12)]

$$A^{(M)}_{M-k+1} \sim a_k(M)\,, \qquad B^{(M)}_{M-k+1} \sim a_0(M-k)\,, \qquad k = 1, \dots, M\,, \tag{2.14}$$

$$B^{(M-1)}_l \sim a_0(l-1) - a_0(M-1)\,, \qquad l = 1, \dots, M-1\,. \tag{2.15}$$

Define now $a_M \equiv A^{(M)}_M \sim a_1(M)$ and $b_M \equiv B^{(M)}_M \sim a_0(M-1)$. As a consequence of the last definition and (2.15), we find the recurrence relation $B^{(M)}_l = b_M + B^{(M-1)}_l$. Furthermore, we notice that identifications made in (2.14) and (2.15) allow us to recast the lattice Toda equation of motion (2.12) in the form of recurrence relations:

$$A^{(M)}_l = A^{(M-1)}_{l-1} + \left(\partial + B^{(M-1)}_l\right)A^{(M-1)}_l\,, \qquad l = 2, \dots, M-1\,. \tag{2.16}$$

Assuming furthermore that $A^{(M)}_0 = a_{M+1}(M)$ we get in addition

$$A^{(M)}_1 = \left(\partial + B^{(M-1)}_1\right)A^{(M-1)}_1\,. \tag{2.17}$$

The above recurrence relations have been shown[8] to be equivalent to the recursive formula for the $2M$-boson KP Lax operator valid for arbitrary $M = 1, 2, 3, \dots$ (with $L_0 \equiv D$, $a_0 \equiv 0$):

$$\begin{aligned} L_M &\equiv L_M(a,b) \equiv L_M(a_1, b_1; \dots; a_M, b_M)\,, \\ L_M &\equiv e^{\int b_M}\left[b_M + (a_M - a_{M-1})D^{-1} + DL_{M-1}D^{-1}\right]e^{-\int b_M}\,. \end{aligned} \tag{2.18}$$

In fact, the solution to (2.18) is the $2M$-field Lax operator of the form of (2.4):[11]

$$L_M = D + \sum_{l=1}^{M} A_l^{(M)} \left(D - B_l^{(M)}\right)^{-1} \left(D - B_{l+1}^{(M)}\right)^{-1} \cdots \left(D - B_M^{(M)}\right)^{-1} \tag{2.19}$$

with coefficients satisfying (2.16) and (2.17). As a result, the latter are expressed in terms of the free boson fields $(a_r, b_r)_{r=1}^{M}$ spanning Heisenberg Poisson bracket algebra

$$\{a_r(x), b_s(y)\}_{P'} = -\delta_{rs}\partial_x \delta(x-y) \tag{2.20}$$

as

$$\begin{aligned} B_l^{(M)} &= \sum_{s=l}^{M} b_s\,, \\ A_M^{(M)} &= a_M\,, \\ A_{M-r}^{(M)} &= \sum_{n_r=r}^{M-1} \cdots \sum_{n_2=2}^{n_3-1} \sum_{n_1=1}^{n_2-1} (\partial + b_{n_r} + \cdots + b_{n_r-r+1}) \cdots \\ &\quad \times (\partial + b_{n_2} + b_{n_2-1})(\partial + b_{n_1}) a_{n_1}\,. \end{aligned} \tag{2.21}$$

This recursive construction of the Lax operator in (2.19) leads to the following proposition:[8]

Proposition. *The $2M$-field Lax operators (2.19) are consistent Poisson reductions of the full KP Lax operator for any $M = 1, 2, 3, \ldots$.*

Thus, the first Poisson bracket structure for L_M from (2.19) is given by

$$\left\{\langle L_M | X\rangle, \langle L_M | Y\rangle\right\}_{P'} = -\langle L_M | [X, Y]\rangle\,, \tag{2.22}$$

where X, Y are arbitrary fixed elements of the algebra of pseudodifferential operators. The subscript P' in (2.22) indicates that the constituents of $L_M(a, b)$ satisfy (2.20).

3. Reductions of the Two-Boson KP Hierarchy to KdV

3.1. *Two-boson KP hierarchy and the Dirac reduction*

We shall consider here truncated elements of the KP hierarchy providing the simplest example of (2.19) and given by Lax operator of the form

$$L_1 = D + a(D-b)^{-1} \tag{3.1}$$

with two free Bose currents (a, b).[16,29] The Lax operator can be cast into the standard form $L_1 = D + \sum_{n=0}^{\infty} w_n D^{-1-n}$, with coefficients $w_n = (-1)^n a(D-b)^n \cdot 1$. A calculation of the Poisson bracket structures using the definition (1.1) and (2.22) yields the first bracket structure of the two-boson (a,b) system: $\{a(x),b(y)\}_1 = -\delta'(x-y)$, and zero otherwise. This implies that the coefficients $w_n(a,b)$ of L_1, as functionals of a, b, satisfy the linear $\mathbf{W}_{1+\infty}$ Poisson bracket algebra. The second bracket structure (1.3) takes in this case the form

$$\begin{aligned}\{a(x),b(y)\}_2 &= -b(x)\delta'(x-y) - \delta''(x-y)\,,\\ \{a(x),a(y)\}_2 &= -2a(x)\delta'(x-y) - a'(x)\delta(x-y)\,,\\ \{b(x),b(y)\}_2 &= -2\delta'(x-y)\,.\end{aligned} \tag{3.2}$$

Now, based on this bracket, $w_n(a,b)$ satisfy the nonlinear $\hat{\mathbf{W}}_\infty$ Poisson bracket algebra.

In Refs. 17 and 30 we applied the Dirac reduction scheme to obtain one-boson KdV hierarchy from the two-boson KP hierarchy. The general feature is a transformation of the two-boson Hamiltonian equations of motion expressed in terms of the second bracket structure $\delta Z/\delta t_r = \{Z, H_r\}_2$ (where Z denotes the original degrees of freedom) to the one-boson Hamiltonian system according to the Dirac scheme:

$$\frac{\partial X}{\partial t_r} = \{X, H_r^D\}_{\mathrm{Dirac}}\,, \tag{3.3}$$

X denoting a surviving one-boson degree of freedom.

Consider the Dirac constraint $\Theta \equiv b = 0$ for the system described by L_1. First, let us discuss the resulting Dirac bracket structure. We find for the surviving variable a

$$\begin{aligned}\{a(x),a(y)\}_2^D &= \{a(x),a(y)\}_2\\ &\quad - \int dz\,dz'\{a(x),\Theta(z)\}_2\{\Theta(z),\Theta(z')\}_2^{-1}\{\Theta(z'),a(y)\}_2\\ &= -\left[2a(x)\partial + a'(x) + \frac{1}{2}\partial^3\right]\delta(x-y)\,.\end{aligned} \tag{3.4}$$

The reduced Lax operator now looks thus:

$$l = D + aD^{-1}\,, \tag{3.5}$$

and the corresponding (nonzero) lowest Hamiltonian functions, $H_r^{\mathrm{KdV}} \equiv \mathrm{Tr}\, l^r/r$, are

$$H_1^{\mathrm{KdV}} = \int a\,,\qquad H_3^{\mathrm{KdV}} = \int a^2\,,\qquad H_5^{\mathrm{KdV}} = \int (2a^3 + aa'')\,. \tag{3.6}$$

Moreover one checks that the flow equation

$$\frac{\delta l}{\delta t_r} = [(l^r)_+, l] \tag{3.7}$$

gives on the lowest level $\delta a/\delta t_1 = a'$ and $\delta a/\delta t_3 = a''' + 6aa'$, the latter equation being the well-known KdV equation.

We shall now demonstrate that the DS reduction is an alternative to the Dirac reduction of the two-boson KP hierarchy to the usual KdV hierarchy.[30]

3.2. *Matrix form of the two-boson KP hierarchy and the DS reduction*

One can associate sl(2) matrices with pseudodifferential Lax operators in the following way:[30]

$$L = D + A + BD^{-1}C \sim \mathcal{A} = \begin{pmatrix} -\frac{1}{2}A & -C \\ B & \frac{1}{2}A \end{pmatrix}, \tag{3.8}$$

so that the gauge transformation of the Lax operator $L' = e^{-\chi}Le^{\chi}$ corresponds to SL(2) gauge transformation $\mathcal{A}' = g\mathcal{A}g^{-1} + g\partial g^{-1}$ with a diagonal 2×2 real unimodular matrix $g = \mathrm{diag}[\exp(\chi/2),\ \exp(-\chi/2)]$.

To see the connection with the matrix Toda hierarchy (with $\lambda = 0$),

$$\begin{pmatrix} \partial - a_0(n-1) & -1 \\ a_1(n) & \partial \end{pmatrix} \begin{pmatrix} \Psi_{n-1} \\ \Psi_n \end{pmatrix} = 0, \tag{3.9}$$

let us introduce new variables as in (2.5):

$$\begin{pmatrix} e^{\frac{1}{2}\int a_0(n-1)}\Psi_{n-1} \\ e^{\frac{1}{2}\int a_0(n-1)}\Psi_n \end{pmatrix}, \tag{3.10}$$

and denote $a_0(n-1) = b$, $a_1(n) = a$. According to (3.8), we find the association

$$L_{\mathrm{KP}} = D + b + aD^{-1} \sim \mathcal{A}_{\mathrm{KP}} = \begin{pmatrix} -\frac{1}{2}b & -1 \\ a & \frac{1}{2}b \end{pmatrix}. \tag{3.11}$$

Here, the important point is that there is a residual gauge transformation generated by

$$g_0 \equiv \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}, \tag{3.12}$$

which preserves the form of $\mathcal{A}_{\mathrm{KP}}$ under

$$\mathcal{A}' = g_0^{-1}\mathcal{A}g_0 + g_0^{-1}\partial g_0 = \begin{pmatrix} -\frac{1}{2}b - \gamma & -1 \\ a + \gamma b + \gamma^2 + \gamma' & \frac{1}{2}b + \gamma \end{pmatrix}. \tag{3.13}$$

Let us analyze what is happening by using the DS formalism. Consider the space of first order differential operators with coefficients being 2×2 matrices:

$$M_{\mathcal{E}} = \left\{ \mathcal{D}^{(1)} = D - \mathcal{E} + \omega \middle| \mathcal{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \omega = \begin{pmatrix} \omega_{11} & 0 \\ \omega_{21} & \omega_{22} \end{pmatrix} \right\}, \tag{3.14}$$

and the group

$$\Gamma \equiv \left\{ \Gamma | \Gamma \equiv \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \right\} , \tag{3.15}$$

acting on $M_{\mathcal{E}}$ according to

$$\Gamma^{-1}(D - \mathcal{E} + \omega)\Gamma = D - \mathcal{E} + \bar{\omega} , \tag{3.16}$$

with

$$\bar{\omega} = \begin{pmatrix} \omega_{11} - \gamma & 0 \\ \omega_{21} + \gamma(\omega_{22} - \omega_{11}) + \gamma^2 + \gamma' & \omega_{22} + \gamma \end{pmatrix} . \tag{3.17}$$

In the spirit of Hamiltonian reduction consider the quotient space $M_{\mathrm{red}} = M_{\mathcal{E}}/\Gamma$. There exists a convenient realization of M_{red} in terms of second order differential operators with scalar coefficients. The procedure for obtaining it goes as follows. Consider the equation

$$\mathcal{D}^{(1)} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 . \tag{3.18}$$

Eliminating ψ_2 from this equation we arrive at $L^{(2)}\psi_1 = 0$ with

$$L^{(2)} \equiv \det(\mathcal{D}^{(1)}) = D^2 + (\omega_{11} + \omega_{22})D + \omega_{21} + \omega_{11}\omega_{22} + \omega_{11}' . \tag{3.19}$$

Clearly $\det(\Gamma^{-1}\mathcal{D}^{(1)}\Gamma) = \det(\mathcal{D}^{(1)})$ and, therefore, the space of the second order differential operators of the form (3.19) parametrizes the quotient space M_{red}.

Let us study now the special case of the two-boson sl(2) matrix:

$$\omega = \begin{pmatrix} \omega_{11} & 0 \\ \omega_{21} & \omega_{22} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}b & 0 \\ a & \frac{1}{2}b \end{pmatrix} . \tag{3.20}$$

For Γ with $\gamma = -\frac{1}{2}b$ the transformed $\bar{\omega}$ matrix

$$\bar{\omega} = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a - \frac{1}{4}b^2 - \frac{1}{2}b' & 0 \end{pmatrix} \tag{3.21}$$

has diagonal elements equal to zero. This means that, according to (3.11), the associated Lax operator is

$$L_{\mathrm{KP}} = D + uD^{-1} , \quad \text{with} \quad u = a - \frac{1}{4}b^2 - \frac{1}{2}b' . \tag{3.22}$$

We can check that, with (a, b) satisfying the second Poisson bracket (3.2), u commutes with b and satisfies the Virasoro algebra:

$$\{u(x), u(y)\} = -2u(x)\delta'(x-y) - u'(x)\delta(x-y) - \frac{1}{2}\delta'''(x-y) . \tag{3.23}$$

We also note that with ω like in (3.20) the second order differential operator (3.19) becomes a typical KdV operator, $L^{(2)} = D^2 + u$. Hence, the first order DS operator

$\mathcal{D}^{(1)}$ [Eq. (3.14)] with $\bar{\omega}$ as in (3.21), or its associated Lax operator $L_{\rm KP}$ from (3.22), represents just a special gauge choice on $M_{\mathcal{E}}$ equivalent to the KdV Lax operator $L^{(2)}$.

3.3. *DS reductions of four-boson KP hierarchy*

Start again with the Toda matrix problem with $\lambda = 0$:

$$\begin{pmatrix} \partial - a_0(n-2) & -1 & 0 \\ 0 & \partial - a_0(n-1) & -1 \\ a_2(n) & a_1(n) & \partial \end{pmatrix} \begin{pmatrix} \Psi_{n-2} \\ \Psi_{n-1} \\ \Psi_n \end{pmatrix} = 0 . \tag{3.24}$$

Consider a general space of first order differential operators with coefficients being 3×3 matrices:

$$M_{\mathcal{E}} = \{\mathcal{D}^{(2)} \equiv D - \mathcal{E} + \omega\} , \tag{3.25}$$

with

$$\mathcal{E} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , \tag{3.26}$$

$$\omega = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ A_1 & A_2 & J_3 \end{pmatrix} . \tag{3.27}$$

Eliminating ψ_1, ψ_2 from the equation

$$\mathcal{D}^{(2)} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = 0 \tag{3.28}$$

we get the Lax operator of the four-boson KP hierarchy,

$$\left(\partial + A_2 \frac{1}{\partial + J_2 - J_3} + A_1 \frac{1}{\partial + J_1 - J_3} \frac{1}{\partial - J_2 - J_3}\right) e^{-\int J_3} \psi_3 = 0 , \tag{3.29}$$

while eliminating ψ_2, ψ_3 we get a KdV type Lax operator:

$$\begin{aligned} &(\partial^3 + u_0 \partial^2 + u_1 \partial + u_2)\psi_1 = 0 , \\ &u_0 \equiv J_1 + J_2 + J_3 , \\ &u_1 \equiv A_2 + 2J_1' + J_2' + J_1 J_2 + J_1 J_3 + J_2 J_3 , \\ &u_2 \equiv A_1 + A_2 J_1 + J_1'' + (J_1 J_2)' + J_1' J_3 + J_1 J_2 J_3 . \end{aligned} \tag{3.30}$$

Applying the same arguments as in Subsec. 3.2, we are led to consider a group consisting of lower-triangular 3×3 matrices,

$$\Gamma \equiv \left\{ \Gamma | \Gamma \equiv \begin{pmatrix} 1 & 0 & 0 \\ \alpha_1 & 1 & 0 \\ \alpha_2 & \alpha_3 & 1 \end{pmatrix} \right\} , \tag{3.31}$$

with an inverse,

$$\Gamma^{-1} \equiv \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_1 & 1 & 0 \\ -\alpha_2 + \alpha_3\alpha_1 & -\alpha_3 & 1 \end{pmatrix} . \tag{3.32}$$

The action of Γ on $M_{\mathcal{E}}$ according to (3.16) produces the following transformation of ω [Eq. (3.27)]:

$$\omega \to \bar{\omega} = \begin{pmatrix} J_1 - \alpha_1 & 0 & 0 \\ \bar{\omega}_{21} & J_2 + \alpha_1 - \alpha_3 & 0 \\ \bar{\omega}_{31} & \bar{\omega}_{32} & J_3 + \alpha_3 \end{pmatrix} , \tag{3.33}$$

where

$$\begin{aligned} \bar{\omega}_{21} &= \alpha_1' + \alpha_1^2 - \alpha_2 + \alpha_1(J_2 - J_1) , \\ \bar{\omega}_{31} &= A_1 + A_2\alpha_1 + \alpha_2' + \alpha_3\alpha_1(J_1 - J_2) \\ &\quad + \alpha_2(\alpha_1 + \alpha_3 + J_3 - J_1) - \alpha_3(\alpha_1^2 + \alpha_1') , \\ \bar{\omega}_{32} &= A_2 + \alpha_3' + \alpha_3(J_3 - J_2 - \alpha_1) + \alpha_2 + \alpha_3^2 . \end{aligned} \tag{3.34}$$

Γ will define the little group preserving the form of $\mathcal{D}^{(2)}$ if we impose $\bar{\omega}_{21} = 0$, i.e.

$$\alpha_2 = \alpha_1' + \alpha_1^2 + \alpha_1(J_2 - J_1) . \tag{3.35}$$

Hence the little group has only two independent components, α_1, α_3, and the transformed matrix $\bar{\omega}$ takes the form

$$\bar{\omega} = \begin{pmatrix} J_1 - \alpha_1 & 0 & 0 \\ 0 & J_2 + \alpha_1 - \alpha_3 & 0 \\ \tilde{A}_1 & \tilde{A}_2 & J_3 + \alpha_3 \end{pmatrix} , \tag{3.36}$$

with

$$\begin{aligned} \tilde{A}_2 &= A_2 + \alpha_1' + \alpha_3' + \alpha_1^2 + \alpha_3^2 + \alpha_1(J_2 - J_1 - \alpha_3) + \alpha_3(J_3 - J_2) , \\ \tilde{A}_1 &= A_1 + \alpha_1'' + (\alpha_1' + \alpha_1^2)(J_2 + J_3 - 2J_1) \\ &\quad + \alpha_1\left[A_2 + \alpha_1^2 + 3\alpha_1' + J_2' - J_1' + (J_2 - J_1)(J_3 - J_1)\right] . \end{aligned} \tag{3.37}$$

Let us return to the Toda problem (3.24), whose matrix differential operator belongs to the space $M_{\mathcal{E}}$ [Eq. (3.25)]. In order to have agreement with future convention, let us introduce the notation

$$\begin{aligned} a_0(n-1) &\equiv B_2 , \qquad a_0(n-2) \equiv B_1 , \\ a_1(n) &\equiv A_2 , \qquad a_2(n) \equiv A_1 . \end{aligned} \tag{3.38}$$

We can again gauge away the trace of the matrix appearing in the spectral problem by letting $\Psi \to \exp\left[-\int(B_2 + B_1)/3\right]\Psi$. This results in a differential operator $\mathcal{D}^{(2)}$ [Eq. (3.25)] with *traceless* matrix ω [Eq. (3.27)], where

$$J_1 = -\frac{2B_1 - B_2}{3} , \qquad J_2 = -\frac{2B_2 - B_1}{3} , \qquad J_3 = \frac{B_2 + B_1}{3} . \tag{3.39}$$

We can choose the two independent parameters α_1, α_3 in Γ [Eq. (3.31)] to eliminate diagonal elements in (3.33) by taking

$$\alpha_1 = -\frac{1}{3}(2B_1 - B_2), \qquad \alpha_3 = -\frac{1}{3}(B_2 + B_1). \tag{3.40}$$

In this case we arrive at the standard DS gauge for $\mathcal{D}^{(2)}$:

$$\bar{\mathcal{D}}^{(2)} = \Gamma^{-1}(D - \mathcal{E} + \omega)\Gamma = \begin{pmatrix} D & -1 & 0 \\ 0 & D & -1 \\ \tilde{A}_1 & \tilde{A}_2 & D \end{pmatrix}, \qquad \det \bar{\mathcal{D}}^{(2)} = \partial^3 + \tilde{A}_2\partial + \tilde{A}_1, \tag{3.41}$$

with

$$\tilde{A}_2 = A_2 - B_1' - \frac{1}{3}(B_2^2 + B_1^2 - B_2B_1),$$
$$\tilde{A}_1 = A_1 - \frac{1}{3}(2B_1'' - B_2'') - \frac{1}{3}(2B_1 - B_2)[A_2 + B_1' - B_2' + (B_1 - B_2)B_1] + \frac{2}{27}(2B_1 - B_2)^3. \tag{3.42}$$

From the second Poisson bracket algebra of the four-boson KP hierarchy, satisfied by $A_{1,2}$, $B_{1,2}$, we find that $\tilde{A}_2$, $\tilde{A}_1$ satisfy the Poisson brackets of Boussinesq hierarchy (the W_3 algebra) (see e.g. Ref. 12)

$$\begin{aligned} \{\tilde{A}_2(x), \tilde{A}_2(y)\}_2 &= -(2\tilde{A}_2\partial + \tilde{A}_2' + 2\partial^3)\delta(x-y), \\ \{\tilde{A}_2(x), \tilde{A}_1(y)\}_2 &= -(3\tilde{A}_1\partial + 2\tilde{A}_1' - \partial^2\tilde{A}_2 - \partial^4)\delta(x-y), \\ \{\tilde{A}_1(x), \tilde{A}_1(y)\}_2 &= -\left[2\tilde{A}_1'\partial + \tilde{A}_1'' - \frac{2}{3}(\tilde{A}_2 + \partial^2)(\partial\tilde{A}_2 + \partial^3)\right]\delta(x-y). \end{aligned} \tag{3.43}$$

The above reduction was complete in the sense that all the diagonal elements were removed. In general, there are plenty of possibilities for a partial reduction with, for example, one diagonal element surviving the reduction. We now describe one such choice. Note first that in the Toda hierarchy problem with a general ω [Eq. (3.27)] the diagonal terms J_i can be represented as

$$J_1 = -B_1 + \bar{B}, \qquad J_2 = -B_2 + \bar{B}, \qquad J_3 = \bar{B}, \tag{3.44}$$

where $\bar{B}$ indicates an addition which can be generated by applying a gauge transformation (which effectively changes the trace of the underlying matrix).

The particular choice of α_1, α_3, $\bar{B}$ we now make is to obtain in (3.36)

$$J_1 - \alpha_1 = \frac{1}{2}(B_2 + B_1), \qquad J_2 + \alpha_1 - \alpha_3 = 0, \qquad J_3 + \alpha_3 = 0. \tag{3.45}$$

The solution is $\alpha_3 = -\bar{B} = -(B_2+B_1)/2$ and $\alpha_1 = -B_1$. Inserting these parameters into (3.37) we get by partial DS reduction a hierarchy with three fields $\tilde{A}_2$, $\tilde{A}_1$, $\mathcal{B} = -(B_2 + B_1)/2$, where

$$\begin{aligned} \tilde{A}_2 &= A_2 - \frac{1}{2}(B_2' + 3B_1') - \frac{1}{4}(B_2 - B_1)^2 \,, \\ \tilde{A}_1 &= A_1 - B_1'' + (B_1B_2)' - B_1A_2 \,. \end{aligned} \tag{3.46}$$

We shall provide below [see (4.7)] the closed Poisson bracket algebra satisfied by these fields reproducing the result obtained in Ref. 12 by a Dirac method.

4. DS Reduction Induced by the KP Poisson Structure

It turns out that the action of the residual gauge symmetry discussed above in the case of two- and four-boson KP hierarchies is Hamiltonian (generated by a Poisson bracket structure). The relevant bracket structure turns out to be the second KP Poisson structure (1.3). Before presenting the general framework, we start with examples of two- and four-boson KP hierarchies to reproduce results of the DS reduction.

4.1. *The case of two bosons*

Recall the second bracket structure for the two-boson system described by (3.2) and consider the Abelian group generated by

$$G = \exp\left(-\int b\beta\right), \tag{4.1}$$

which acts on (a, b) through the second bracket. For example

$$G^{-1}b(x)G = b(x) - \left\{b(x), \int b\beta\right\}_2 + \frac{1}{2}\left\{\left\{b(x), \int b\beta\right\}_2, \int b\beta\right\}_2 + \cdots \tag{4.2}$$

A simple calculation shows that the result of such an action on (a, b) is

$$\begin{aligned} b &\to G^{-1}b(x)G = b + 2\beta' \,, \\ a &\to G^{-1}a(x)G = a + b\beta' + \beta'' + (\beta')^2 \,. \end{aligned} \tag{4.3}$$

Choosing $\beta' = \gamma$ we see that (4.3) reproduces the result in (3.13), as we set out to show. Since this transformation is generated by the current $b(x)$ satisfying Heisenberg Poisson bracket algebra [last of Eqs. (3.2)], we see that transforming $b \to 0$ by the above transformation amounts to imposition of the Dirac constraint $b = 0$. This explains why the above DS gauge choice in (3.21) leading to KdV was equivalent to taking the Dirac bracket. This explanation is based on the curious equivalence between transformations generated by lower-triangular matrices acting on matrices representing KP hierarchy and transformations generated by currents acting on the second bracket Poisson manifold of the multiboson KP hierarchy.

Note that the quantity $u = a - \frac{1}{4}b^2 - \frac{1}{2}b'$ parametrizing the reduced manifold is invariant under the transformation (4.3) since $\{u, b\} = 0$, the point which is obvious in matrix formulation but which can only be verified *a posteriori* for the current-generated transformations. This fact is crucial for explaining why from algebraic point of view the Dirac conditions $b = 0$, $a = u$ agree with the gauge-fixing as done above. We shall see that this relationship is of a general nature and applies to arbitrary multiboson KP systems.

4.2. *The case of four bosons*

In the case of four-boson KP hierarchy consider a group action on the KP Poisson manifold generated by

$$G = \exp\left\{ - \int [B_2(\gamma_3 - \gamma_1) + B_1\gamma_1]\right\} . \tag{4.4}$$

Our choice of the γ parameters is dictated by the conventions of the corresponding DS reduction, as shown above. The action of G gives [e.g. $G^{-1}B_2G$; cf. (4.2)]

$$\begin{aligned} B_2 &\to B_2 + 2\gamma_3' - \gamma_1' , \\ B_1 &\to B_1 + \gamma_3' + \gamma_1' , \\ A_2 &\to A_2 + \gamma_3'' + \gamma_1'' + B_2(\gamma_3' - \gamma_1') + B_1\gamma_1' + (\gamma_3')^2 + (\gamma_1')^2 - \gamma_1'\gamma_3' , \\ A_1 &\to A_1 + A_2\gamma_1' + \gamma_1''' + (2B_1 - B_2)\gamma_1'' + (B_1' - B_2')\gamma_1' \\ &\quad + B_1(B_1 - B_2)\gamma_1' + 3\gamma_1'\gamma_1'' + (2B_1 - B_2)\gamma_1'\gamma_1' + (\gamma_1')^3 . \end{aligned} \tag{4.5}$$

Taking $\gamma_i' = \alpha_i$ we recover (3.37) with the choice of B_i's as in (3.39). We can now eliminate B_2 and B_1 by appropriate choice of α's recovering the Boussinesq hierarchy (3.42).

Now, let us discuss the problem of invariance under α_1, α_3 transformations on the reduced manifold. First, we notice that $\tilde{A}_1$, $\tilde{A}_2$ of the Boussinesq hierarchy (3.42) are invariant under transformations given in (4.5). The situation with partial gauge-fixing is more subtle. We shall examine the invariance under separate α_1 and α_3 transformations by using the first two relations from (4.5). There are two possibilities:

(1) $B_2 + B_1$ — invariant under α_1 transformation;
(2) $2B_1 - B_2$ — invariant under α_3 transformation.

In case (1) we get, as before, three fields $\tilde{A}_1$, $\tilde{A}_2$, $\mathcal{B} = -(B_2 + B_1)/2$, where A_1, A_2 are given by (3.46) — both being invariant under α_1 transformation. This can be achieved by choosing (for instance) $\alpha_3 = -(B_2 + B_1)/2$ and $\alpha_1 = -B_1$ so $B_2 \to 0$ and $B_1 \to -(B_2 + B_1)/2$.

In case (2) we take $\alpha_1 = 0$, $\alpha_3 = -\frac{1}{2}B_2$ in order to have transformations $B_2 \to 0$ and $B_1 \to \mathcal{B} \equiv \frac{1}{2}(2B_1 - B_2)$. With this choice we get

$$\tilde{A}_2 = A_2 - \frac{1}{2}B_2' - \frac{1}{4}B_2^2\,, \qquad \tilde{A}_1 = A_1\,, \tag{4.6}$$

which could be obtained by putting $B_1 = 0$ in (3.46). It is easy to see that (4.6) is invariant under α_3 transformation. One easily verifies that for $A_{1,2}$, $B_{1,2}$ satisfying the Poisson brackets dictated by (1.3), the fields (4.6) together with $\mathcal{B}$ satisfy a closed algebra:

$$\begin{aligned}
\{\tilde{A}_2(x), \tilde{A}_2(y)\}_2 &= -\left(2\tilde{A}_2\partial + \tilde{A}_2' + \frac{1}{2}\partial^3\right)\delta(x-y)\,,\\
\{\tilde{A}_2(x), \tilde{A}_1(y)\}_2 &= -(3\tilde{A}_1\partial + 2\tilde{A}_1')\delta(x-y)\,,\\
\{\tilde{A}_2(x), \mathcal{B}(y)\}_2 &= -\left(\frac{3}{2}\partial^2 + \mathcal{B}\partial\right)\delta(x-y)\,,\\
\{\tilde{A}_1(x), \tilde{A}_1(y)\}_2 &= -[(2\tilde{A}_1' + 4\tilde{A}_1\mathcal{B})\partial + \tilde{A}_1'' + 2(\tilde{A}_1\mathcal{B})']\delta(x-y)\,,\\
\{\tilde{A}_1(x), \mathcal{B}(y)\}_2 &= -[\tilde{A}_2\partial + (\partial + \mathcal{B})^2\partial]\delta(x-y)\,,\\
\{\mathcal{B}(x), \mathcal{B}(y)\}_2 &= -\frac{3}{2}\delta'(x-y)\,,
\end{aligned} \tag{4.7}$$

obtained first by the Dirac bracket method in Ref. 12.

5. Generalized Miura Transformation for Multiboson KP Hierarchies

The two-boson hierarchy given in Sec. 3 and described by L_1 [Eq. (3.1)] is equivalent to the model based on the pseudodifferential operator:[31]

$$L_1 = (D-e)(D-c)(D-e-c)^{-1} = D + (e' + ec)(D-e-c)^{-1}\,. \tag{5.1}$$

The Miura-like connection between these hierarchies generalizes the usual Miura transformation between one-Bose KdV and mKdV structures and takes the form[17]

$$a = e' + ec\,, \qquad b = e + c\,. \tag{5.2}$$

This Miura transformation, $(e, c) \to (a, b)$, can easily be seen to Abelianize the second bracket (3.2), meaning that whenever

$$\{e(x), c(y)\}_2 = -\delta'(x-y) \tag{5.3}$$

a, b, given by (5.2), satisfy (3.2).

The above structures naturally appear in connection with the Toda and Volterra lattice hierarchies.[15] Consider namely the spectral equation

$$\partial\Psi_n = \Psi_{n+1} + a_0(n)\Psi_n\,, \qquad \lambda\Psi_n = \Psi_{n+1} + a_0(n)\Psi_n + a_1(n)\Psi_{n-1}\,. \tag{5.4}$$

The Miura-transformed hierarchy defined by (5.1) can be associated with a "square root" lattice with respect to the original Toda lattice system (5.4):

$$\begin{aligned} \lambda^{1/2}\tilde{\Psi}_{n+\frac{1}{2}} &= \Psi_{n+1} + \mathcal{A}_{n+1}\Psi_n\,, & \lambda^{1/2}\Psi_n &= \tilde{\Psi}_{n+\frac{1}{2}} + \mathcal{B}_n\tilde{\Psi}_{n-\frac{1}{2}}\,, \\ \tilde{\Psi}_{n+\frac{1}{2}} &= (\partial - \mathcal{B}_n - \mathcal{A}_n)\tilde{\Psi}_{n-\frac{1}{2}}\,, & \Psi_{n+1} &= (\partial - \mathcal{B}_n - \mathcal{A}_{n+1})\Psi_n\,, \end{aligned} \tag{5.5}$$

which yields the Volterra chain equations.[15] Excluding the half-integer modes in (5.5) we recover (5.4) with

$$a_0(n) = \mathcal{A}_{n+1} + \mathcal{B}_n\,, \qquad a_1(n) = \mathcal{A}_n\mathcal{B}_n = \mathcal{B}_{n-1} + \partial\mathcal{A}_n\,, \tag{5.6}$$

where in the latter equation we have used one of the Volterra equations of motion [consistency condition for the Volterra spectral problem (5.5)]:

$$\partial\mathcal{A}_n = \mathcal{A}_n(\mathcal{B}_n - \mathcal{B}_{n-1})\,. \tag{5.7}$$

Equations (5.5) can be cast into the form

$$\lambda\Psi_n = L_n^{(1)}\Psi_n\,, \qquad L_n^{(1)} = (\partial - \mathcal{A}_n)(\partial - \mathcal{B}_{n-1})(\partial - \mathcal{B}_{n-1} - \mathcal{A}_n)^{-1}\,, \tag{5.8}$$

which, upon the identification $\mathcal{A}_n = e$, $\mathcal{B}_{n-1} = c$, agrees with (5.1). Moreover, recalling the identification $a = a_1(n)$, $b = a_0(n-1)$ for the coefficients of L_1 [Eq. (3.1)] [cf. (2.4)], we see that the relation (5.6) between the coefficients of the Toda and Volterra discrete spectral problems precisely matches the Miura relation (5.2) for the coefficients of the two-boson KP Lax operators (3.1) and (5.1) Abelianizing the first and second KP Poisson bracket structures, respectively.

Furthermore, using again the Volterra equation (5.7) we can rewrite (5.8) as

$$L_n^{(1)} = (\partial - \mathcal{A}_n)(\partial - \mathcal{B}_n - \mathcal{A}_n)^{-1}(\partial - \mathcal{B}_n) = \partial + \mathcal{B}_n(\partial - \mathcal{B}_n - \mathcal{A}_n)^{-1}\mathcal{A}_n\,, \tag{5.9}$$

which, upon the identification $\mathcal{B}_n{=}\bar{\jmath}$, $\mathcal{A}_n{=}\jmath$, takes the form $L{=}D{+}\bar{\jmath}\,(D{-}\jmath{-}\bar{\jmath})^{-1}\jmath$. This is the form of the two-boson KP hierarchy which appeared in connection with the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model.[31]

The above simple two-boson model will now be generalized to the arbitrary multiboson KP hierarchies.

We start by finding a "square root" lattice formulation corresponding to the general Toda lattice hierarchy (2.1). In this spirit we are led to a spectral equation:

$$\lambda^{1/2}\tilde{\Psi}_{n+\frac{1}{2}} = \Psi_{n+1} + \mathcal{A}^{(0)}_{n+1}\Psi_n + \sum_{p=1}^{M}\mathcal{A}^{(p)}_{n-p+1}\Psi_{n-p}\,, \tag{5.10}$$

$$\lambda^{1/2}\Psi_n = \tilde{\Psi}_{n+\frac{1}{2}} + \mathcal{B}^{(0)}_n\tilde{\Psi}_{n-\frac{1}{2}}\,, \tag{5.11}$$

with time evolution equations:

$$\begin{aligned}\tilde{\Psi}_{n+\frac{1}{2}} &= \left(\partial - \mathcal{B}_n^{(0)} - \mathcal{A}_n^{(0)}\right)\tilde{\Psi}_{n-\frac{1}{2}}\,,\\ \Psi_{n+1} &= \left(\partial - \mathcal{B}_n^{(0)} - \mathcal{A}_{n+1}^{(0)}\right)\Psi_n\,.\end{aligned} \tag{5.12}$$

As in the two-boson case above, it is straightforward to show that, upon excluding the half-integer modes, the generalized Volterra system (5.10)–(5.12) reduces to the Toda lattice spectral equations (2.1) for $M+1$, where

$$\begin{aligned} a_0(n) &= \mathcal{A}_{n+1}^{(0)} + \mathcal{B}_n^{(0)}\,, \qquad a_{M+1}(n) = \mathcal{B}_n^{(0)}\mathcal{A}_{n-M}^{(M)}\,,\\ a_p(n) &= \mathcal{A}_{n-p+1}^{(p)} + \mathcal{B}_n^{(0)}\mathcal{A}_{n-p+1}^{(p-1)}\,, \qquad p=1,\dots,M\,.\end{aligned} \tag{5.13}$$

From (5.10)–(5.12) we find that

$$\lambda^{1/2}\tilde{\Psi}_{n+\frac{1}{2}} = \left[\partial - \mathcal{B}_n^{(0)} + \sum_{p=1}^{M}\mathcal{A}_{n-p+1}^{(p)}\left(\partial - \mathcal{B}_{n-p}^{(0)} - \mathcal{A}_{n-p+1}^{(0)}\right)^{-1}\cdots\left(\partial - \mathcal{B}_{n-1}^{(0)} - \mathcal{A}_n^{(0)}\right)^{-1}\right]\Psi_n\,, \tag{5.14}$$

$$\lambda^{1/2}\Psi_n = \left(\partial - \mathcal{A}_n^{(0)}\right)\tilde{\Psi}_{n-\frac{1}{2}}\,. \tag{5.15}$$

From the last two relations it follows that

$$\begin{aligned}\lambda\Psi_n = \left(\partial - \mathcal{A}_n^{(0)}\right)&\left[\partial - \mathcal{B}_{n-1}^{(0)} + \sum_{p=1}^{M}\mathcal{A}_{n-p}^{(p)}\left(\partial - \mathcal{B}_{n-p-1}^{(0)} - \mathcal{A}_{n-p}^{(0)}\right)^{-1}\cdots\left(\partial - \mathcal{B}_{n-2}^{(0)} - \mathcal{A}_{n-1}^{(0)}\right)^{-1}\right]\\ &\times\left(\partial - \mathcal{B}_{n-1}^{(0)} - \mathcal{A}_n^{(0)}\right)^{-1}\Psi_n\,.\end{aligned} \tag{5.16}$$

This defines a Lax operator through $\lambda\Psi_n = L_n^{(M+1)}\Psi_n$, where

$$L_n^{(M+1)} = e^{\int \mathcal{B}_{n-1}^{(0)}}\left(\partial - \mathcal{A}_n^{(0)} + \mathcal{B}_{n-1}^{(0)}\right)L_n^{(M)}\left(\partial - \mathcal{A}_n^{(0)}\right)^{-1}e^{-\int \mathcal{B}_{n-1}^{(0)}}\,, \tag{5.17}$$

$$\begin{aligned} L_n^{(M)} = \partial + \sum_{p=1}^{M}\mathcal{A}_{n-p}^{(p)}&\left(\partial + \mathcal{B}_{n-1}^{(0)} - \mathcal{B}_{n-p-1}^{(0)} - \mathcal{A}_{n-p}^{(0)}\right)^{-1}\cdots\\ &\times\left(\partial + \mathcal{B}_{n-1}^{(0)} - \mathcal{B}_{n-2}^{(0)} - \mathcal{A}_{n-1}^{(0)}\right)^{-1}.\end{aligned} \tag{5.18}$$

Equations (5.17)–(5.18) can be identified with the recurrence relation for the $2M$-boson KP Lax operators (2.19) established in Ref. 20:

$$\begin{aligned} L_{M+1} &= e^{\int c_{M+1}}(D + c_{M+1} - e_{M+1})L_M(D - e_{M+1})^{-1}e^{-\int c_{M+1}}\,,\\ &\qquad M = 0,1,2,\dots\,, \qquad L_0 \equiv D\,,\end{aligned} \tag{5.19}$$

$$\{c_k(x), e_l(y)\} = -\delta_{kl}\partial_x\delta(x-y)\,, \qquad k,l = 1,2,\dots,M+1\,. \tag{5.20}$$

The free field pairs $(c_r, e_r)_{r=1}^{M+1}$ are the "Darboux–Poisson" canonical pairs for the second KP bracket (1.3) satisfied by L_{M+1} for arbitrary M.[20] This defines a sequence of the multiboson KP Lax operators in terms of Darboux–Poisson free field pairs with respect to the second KP bracket, very much as (2.18) defined a similar sequence of Lax operators in terms of Darboux–Poisson free field pairs with respect to the first KP bracket.[8] This construction can be viewed as a generalized Miura transformation for multiboson KP hierarchies, and hence "Abelianization" of the *second* KP Hamiltonian structure (1.3), i.e. expressing the coefficient fields of the pertinent KP Lax operator in terms of canonical pairs of free fields.

Equation (5.19) implies the following recurrence relations for the coefficient fields of L_M [Eq. (2.19)] (see also Ref. 20):

$$B_k^{(M+1)} = B_k^{(M)} + c_{M+1}\,, \quad 1 \le k \le M\,, \quad B_{M+1}^{(M+1)} = c_{M+1} + e_{M+1}\,, \tag{5.21}$$

$$A_1^{(M+1)} = \left(\partial + B_1^{(M)} + c_{M+1} - e_{M+1}\right) A_1^{(M)}\,, \tag{5.22}$$

$$A_k^{(M+1)} = A_{k-1}^{(M)} + \left(\partial + B_k^{(M)} + c_{M+1} - e_{M+1}\right) A_k^{(M)}\,, \quad 2 \le k \le M\,, \tag{5.23}$$

$$A_{M+1}^{(M+1)} = A_M^{(M)} + (\partial + c_{M+1}) e_{M+1}\,. \tag{5.24}$$

As found in Ref. 20, the recurrence relations (5.21)–(5.24) have an explicit solution:

$$B_k^{(M)} = e_k + \sum_{l=k}^{M} c_l\,, \qquad 1 \le k \le M\,, \qquad A_M^{(M)} = \sum_{k=1}^{M} (\partial + c_k) e_k\,, \tag{5.25}$$

$$\begin{aligned} A_k^{(M)} = & \sum_{n_{M-k+1}=1}^{k} \left(\partial + e_{n_{M-k+1}} - e_{n_{M-k+1}+M-k} + \sum_{l_k = n_{M-k+1}}^{n_{M-k+1}+M-k} c_{l_k} \right) \\ & \times \sum_{n_{M-k}=1}^{n_{M-k+1}} \left(\partial + e_{n_{M-k}} - e_{n_{M-k}+M-1-k} + \sum_{l_{k-1} = n_{M-k}}^{n_{M-k}+M-1-k} c_{l_{k-1}} \right) \times \cdots \\ & \times \sum_{n_2=1}^{n_3} (\partial + e_{n_2} - e_{n_2+1} + c_{n_2} + c_{n_2+1}) \\ & \times \sum_{n_1=1}^{n_2} (\partial + c_{n_1}) e_{n_1}\,, \qquad k = 1, \ldots, M-1\,, \end{aligned} \tag{5.26}$$

in terms of the free fields $(c_r, e_r)_{r=1}^{M}$.

A simple calculation, based on the explicit expressions (5.25) and (5.26), gives the following general relations (valid for any M):

Proposition

$$\begin{aligned}
&\{B_i^{(M)}(x), B_j^{(M)}(y)\} = -X_{ij}\partial_x\delta(x-y)\,, \qquad X_{ij} \equiv \delta_{ij} + 1\,,\\
&\{A_M^{(M)}(x), B_k^{(M)}(y)\} = -\left[(M+1-k)\partial_x + B_k^{(M)}\right]\partial_x\delta(x-y)\,, \qquad 1 \le k \le M\,,\\
&\{A_M^{(M)}(x), A_M^{(M)}(y)\} = -\left[A_M^{(M)}(x)\partial_x + \partial_x A_M^{(M)}\right]\delta(x-y)\,,\\
&\{A_i^{(M)}(x), B_j^{(M)}(y)\} = 0\,, \qquad i < j\,,
\end{aligned} \tag{5.27}$$

where for brevity we have recorded only the most simple relations following from (5.25) and (5.26).

(1) *Example* — two-boson KP:

$$\begin{aligned}
L_1 &= e^{\int c_1}(D + c_1 - e_1)D(D - e_1)^{-1}e^{-\int c_1}\\
&= D + A_1^{(1)}\left(D - B_1^{(1)}\right)^{-1},
\end{aligned} \tag{5.28}$$

$$A_1^{(1)} = (\partial + c_1)e_1\,, \qquad B_1^{(1)} = c_1 + e_1\,. \tag{5.29}$$

Here we recognize the structure of the two-boson hierarchy from (5.1) as well as the generalized Miura map (5.2).

(2) *Example* — four-boson KP:

$$\begin{aligned}
L_2 &= e^{\int c_2}(D + c_2 - e_2)\left[D + A_1^{(1)}\left(D - B_1^{(1)}\right)^{-1}\right](D - e_2)^{-1}e^{-\int c_2}\\
&= D + A_2^{(2)}\left(D - B_2^{(2)}\right)^{-1} + A_1^{(2)}\left(D - B_1^{(2)}\right)^{-1}\left(D - B_2^{(2)}\right)^{-1},
\end{aligned} \tag{5.30}$$

$$A_2^{(2)} = A_1^{(1)} + (\partial + c_2)e_2 = (\partial + c_1)e_1 + (\partial + c_2)e_2\,, \tag{5.31}$$

$$\begin{aligned}
A_1^{(2)} &= \left(\partial + B_1^{(1)} + c_2 - e_2\right)A_1^{(1)}\\
&= (\partial + e_1 + c_1 + c_2 - e_2)(\partial + c_1)e_1\,,
\end{aligned} \tag{5.32}$$

$$B_2^{(2)} = c_2 + e_2\,, \qquad B_1^{(2)} = B_1^{(1)} + c_2 = e_1 + c_1 + c_2\,, \tag{5.33}$$

where $A_1^{(1)}$ and $B_1^{(1)}$ have been substituted with their expressions (5.29). It is easy to derive the second bracket structure for the above fields directly from (5.20).

From the recursive relation (5.19) we can obtain closed expressions for the general Lax operator L_M, $M = 1, 2, \dots$, directly in terms of the building blocks $(c_k, e_k)_{k=1}^{M}$:

$$L_M = (D - e_M) \prod_{k=M-1}^{1} \left(D - e_k - \sum_{l=k+1}^{M} c_l \right)$$

$$\times \left(D - \sum_{l=1}^{M} c_l \right) \prod_{k=1}^{M} \left(D - e_k - \sum_{l=k}^{M} c_l \right)^{-1}$$

$$= \prod_{k=M}^{1} (D + c_k - B_k) \left(D - \sum_{l=1}^{M} c_l \right) \prod_{k=1}^{M} (D - B_k)^{-1} ; \qquad (5.34)$$

for brevity we drop from now on the superscript M, so that $B_k \equiv B_k^{(M)} = e_k + \sum_{l=k}^{K} c_l$. Let us introduce now new linear combinations:

$$\bar{B}_i \equiv c_{M-i+1} - B_{M-i+1} , \qquad \bar{B}_{M+1} \equiv - \sum_{l=1}^{M} c_l , \qquad i = 1, \dots , M . \qquad (5.35)$$

The advantage of this notation is that the Lax operator L_M [Eq. (5.34)] takes a more compact form:

$$L_M = \prod_{j=1}^{M+1} (D + \bar{B}_j) \prod_{k=1}^{M} (D - B_k)^{-1} . \qquad (5.36)$$

This form of the Lax operator has already appeared in Ref. 21. The fields (currents) $\bar{B}_j$, B_i satisfy by definition the condition

$$\psi_{M+1,M} \equiv \sum_{j=1}^{M+1} \bar{B}_j + \sum_{k=1}^{M} B_k = 0 , \qquad (5.37)$$

recognized in Ref. 21 as a tracelessness condition of the graded SL$(M+1, M)$ Kac–Moody algebra. It follows from (5.27) and from

$$\{c_i(x), B_j(y)\} = -\delta_{ij} \delta'(x-y) \qquad (5.38)$$

that the Poisson bracket algebra satisfied by the fields $\bar{B}_j$, B_i is the Cartan subalgebra of the graded SL$(M+1, M)$ Kac–Moody algebra:

$$\begin{aligned} \{\bar{B}_i(x), \bar{B}_j(y)\} &= (\delta_{ij} - 1) \delta'(x-y) , \qquad i,j = 1, \dots , M+1 , \\ \{B_k(x), B_l(y)\} &= -(\delta_{kl} + 1) \delta'(x-y) , \qquad k,l = 1, \dots , M , \\ \{\bar{B}_i(x), B_l(y)\} &= \delta'(x-y) . \end{aligned} \qquad (5.39)$$

We shall refer to the algebra (5.39) as SL$_c(M+1, M)$ Kac–Moody algebra.

6. Reduction of SL$(M+1, M)$ to SL$(M+1, M-k)$

Here we present a general scheme for gauging away k (out of M) currents $B_{M-k+1}, \dots , B_M$ by introducing the gauge generator

$$G = \exp\left(-\int\sum_{i=1}^{M} B_i\gamma_i\right), \tag{6.1}$$

which induces the following transformations via Hamiltonian action [as in (4.2)]:

$$\begin{aligned} B_i &\to \tilde{B}_i = G^{-1}B_iG = B_i + X_{ij}\gamma_j', \\ c_i &\to \tilde{c}_i = G^{-1}c_iG = c_i + \gamma_i', \\ e_i &\to \tilde{e}_i = G^{-1}e_iG = e_i + \sum_{l=1}^{i}\gamma_l'. \end{aligned} \tag{6.2}$$

Note that γ_i' are fixed by the gauge-fixing condition $\tilde{B}_i = 0$, $i = M - k + 1, \dots, M$, i.e.

$$\gamma_i' = -X_{ij}^{(k)^{-1}}B_j, \qquad X_{ij}^{(k)} \equiv 1 + \delta_{ij}, \tag{6.3}$$

$X_{ij}^{(k)}$ being the restriction of X_{ij} to $M - k + 1 \le i,\, j \le M$ and $\gamma_r' = 0$ for $1 \le r \le M - k$.

In what follows we shall need to find explicitly the inverse of the matrix $X_{ij} \equiv 1 + \delta_{ij}$. Let $\mathbf{U}$ be an $M \times M$ matrix with elements $U_{ij} = 1$. Hence $\mathbf{X} = \mathbb{1} + \mathbf{U}$ and it is easy to see that

$$\begin{aligned} \mathbf{X}^{-1} &= \mathbb{1} - \mathbf{U} + \mathbf{U}^2 - \cdots \\ &= \mathbb{1} - \mathbf{U}(1 - M + M^2 - M^3 + \cdots) \\ &= \mathbb{1} - \frac{1}{1+M}\mathbf{U}. \end{aligned} \tag{6.4}$$

Correspondingly we also find that

$$\mathbf{X}^{(\mathbf{k})^{-1}} = \mathbb{1} - \frac{1}{1+k}\mathbf{U}^{(\mathbf{k})}, \tag{6.5}$$

where again the superscript (k) indicates restriction of the matrix indices to $M - k + 1 \le i,\, j \le M$.

With this information we can rewrite (6.3) as

$$\gamma_i' = -B_i + \frac{1}{k+1}\sum_{n=M-k+1}^{M} B_n, \qquad M - k + 1 \le i \le M. \tag{6.6}$$

From (6.6) and (6.2) we find the values of the gauge-rotated nonzero B_i to be given by

$$\tilde{B}_r = B_r - \frac{1}{k+1}\sum_{n=M-k+1}^{M} B_n, \qquad 1 \le r \le M - k. \tag{6.7}$$

For the gauge-transformed c_i we find that

$$\tilde{c}_i = \begin{cases} c_i \,, & 1 \le i \le M-k \,, \\ c_i - B_i + \dfrac{1}{k+1} \displaystyle\sum_{n=M-k+1}^{M} B_n \,, & M-k+1 \le i \le M \,. \end{cases} \tag{6.8}$$

Correspondingly, we obtain for the new gauge-transformed $\bar{B}_i = \left(\tilde{c}_{M-i+1} - \tilde{B}_{M-i+1}, -\sum_{l=1}^{M} \tilde{c}_l\right)$ [see (5.35)] (omitting the tilde on top of $\bar{B}_i$ for brevity)

$$\bar{B}_i = \begin{cases} c_{M-i+1} - B_{M-i+1} + \dfrac{1}{k+1} \displaystyle\sum_{n=M-k+1}^{M} B_n \,, & 1 \le i \le M \,, \\ -\displaystyle\sum_{l=1}^{M} c_l + \dfrac{1}{k+1} \displaystyle\sum_{n=M-k+1}^{M} B_n \,, & i = M+1 \,. \end{cases} \tag{6.9}$$

Using (5.27) and (5.38) we find that $\bar{B}_i$, $\tilde{B}_r$ satisfy the Poisson bracket Cartan subalgebra of the graded SL($M+1$, $M-k$) Kac–Moody algebra [cf. Eqs. (5.39)]:

$$\begin{aligned} \{\bar{B}_i(x), \bar{B}_j(y)\} &= \left(\delta_{ij} - \frac{1}{k+1}\right)\delta'(x-y)\,, \qquad i,j = 1,\dots, M+1\,, \\ \{\tilde{B}_r(x), \tilde{B}_s(y)\} &= -\left(\delta_{rs} + \frac{1}{k+1}\right)\delta'(x-y)\,, \qquad 1 \le r,\ s \le M-k\,, \\ \{\bar{B}_i(x), \tilde{B}_r(y)\} &= -\frac{1}{k+1}\,\delta'(x-y)\,, \end{aligned} \tag{6.10}$$

for which we shall use the symbol $\mathrm{SL}_c(M+1, M-k)$. Note that the gauge transformation (6.2) maps the trace constraint $\psi_{M+1,M}$ [see (5.37)] to the new SL($M+1$, $M-k$) trace condition:

$$\psi_{M+1,M-k} \equiv G^{-1}\psi_{M+1,M}G = \sum_{j=1}^{M+1} \bar{B}_j + \sum_{r=1}^{M-k} \tilde{B}_r = 0\,. \tag{6.11}$$

In addition to $\bar{B}_i$, $\tilde{B}_r$ there are k gauge parameters γ'_n [Eq. (6.6)] associated with the SL($k+1$) algebra and satisfying

$$\begin{gathered} \{\gamma'_n(x), \gamma'_m(y)\} = -\left(\delta_{nm} - \frac{1}{k+1}\right)\delta'(x-y)\,, \\ n,m = M-k+1,\dots, M\,, \end{gathered} \tag{6.12}$$

which we shall call $\mathrm{SL}_c(k+1)$ algebra. Note that γ'_n are decoupled from $\bar{B}_i$, $\tilde{B}_r$:

$$\begin{gathered} \{\gamma'_n, \bar{B}_i\} = \{\gamma'_n, \tilde{B}_r\} = 0\,, \\ n = M-k+1,\dots, M\,, \qquad r = 1,\dots, M-k\,, \qquad i = 1,\dots, M+1\,. \end{gathered} \tag{6.13}$$

We therefore easily arrive at the following proposition:

Proposition. *The second bracket of the multiboson KP is reducible under the gauge-fixing procedure described above and* $\{\bar{B}_i, B_n\} = \{\tilde{B}_r, B_n\} = 0$, *where* B_n, $M-k+1 \le n \le M$, *are the modes gauged away.*

This result generalizes the observation we made in Subsec. 3.2, concerning the decoupling of the KdV mode $u = a - \frac{1}{4}b^2 - \frac{1}{2}b'$ from the current b.

The above process of reduction can be extended to remove all currents, i.e. $\tilde{B}_i = 0$ for $i = 1, \ldots, M$. In this case

$$\gamma_i' = -X_{ij}^{-1} B_j , \qquad \tilde{c}_i = c_i - X_{ij}^{-1} B_j . \tag{6.14}$$

Accordingly $\bar{B}_i = \left(\tilde{c}_{M-i+1}, -\sum_{l=1}^{M} c_l\right)$ and the underlying Poisson bracket algebra splits into two disjoint $\mathrm{SL}_c(M+1)$ algebras of opposite signatures:

$$\begin{aligned} \{\bar{B}_i(x), \bar{B}_j(y)\} &= \left(\delta_{ij} - \frac{1}{1+M}\right)\delta'(x-y) , \qquad i,j = 1, \ldots, M , \\ \{\gamma_i'(x), \gamma_j'(y)\} &= -\left(\delta_{ij} - \frac{1}{1+M}\right)\delta'(x-y) , \qquad i,j = 1, \ldots, M , \\ \{\bar{B}_i(x), \gamma_j'(y)\} &= 0 . \end{aligned} \tag{6.15}$$

Example — two-boson KP

Recall the expressions (5.2), $A = e' + ce$, $B = c + e$. In this case $\tilde{B} = B + 2\gamma'$, $\tilde{c} = c + \gamma'$ and the condition $\tilde{B} = 0$ leads to $\tilde{c} = (c-e)/2$ and $\gamma' = -(c+e)/2$. Correspondingly we find that $A \to \tilde{A} = \tilde{e}' + \tilde{c}\tilde{e} = (e-c)'/2 - (e-c)^2/4$, which satisfies the Virasoro algebra due to $\{(e-c)/2,\ (e-c)/2\} = -\delta'(x-y)/2$. Note also that we can rewrite $\tilde{A}$ as $A - \frac{1}{4}B^2 - \frac{1}{2}B'$, obtaining agreement with the result of the DS reduction (3.22).

The above considerations show that the graded $\mathrm{SL}_c(M+1, M)$ algebra is reducible and the splitting is of the following form:

Proposition. $\mathrm{SL}_c(M+1, M) = \mathrm{SL}_c(M+1, M-k) \oplus \mathrm{SL}_c(k+1)$ *with* $k = 1, \ldots, M$.

Indeed, given $\{\bar{\beta}_i, \beta_r\}$ with $1 \le i \le M+1$, $1 \le r \le M-k$ spanning $\mathrm{SL}_c(M+1, M-k)$ and an independent basis $\{\alpha_n\}$ for $\mathrm{SL}_c(k+1)$ with $M-k+1 \le n \le M$, we can form an $\mathrm{SL}_c(M+1, M)$ algebra by taking the linear combination

$$\bar{B}_i = \begin{cases} \bar{\beta}_n - \alpha_n + X_{nm}^{(k)}\alpha_m , \\ \beta_r + X_{rm}\alpha_m , \\ \bar{\beta}_{M+1} + \sum_{n=1}^{k} \alpha_n , \end{cases} \qquad B_i = \begin{cases} -X_{nm}^{(k)}\alpha_m , \\ \beta_r - X_{rm}\alpha_m . \end{cases} \tag{6.16}$$

Note that this construction, as we have seen above, is reversible. Note also that for $k = M$ we have a decomposition into two independent $\mathrm{SL}_c(M+1)$ algebras of opposite signatures, as in (6.15).

6.1. *The second bracket structure of the* SL$(M+1, M-k)$ *KdV hierarchy*

Let us recall the expression (6.10) for the graded Poisson bracket algebra $\mathrm{SL}_c(M+1, M-k)$. In Ref. 21 this algebra was realized as a Dirac bracket algebra obtained from the Poisson brackets of two independent sets of Bose fields of opposite signatures,

$$\begin{aligned} &\{\bar{B}_i(x), \bar{B}_j(y)\}_{\mathrm{PB}} = \delta_{ij}\delta'(x-y)\,, \qquad i,j = 1,\dots, M+1\,, \\ &\{B_m(x), B_n(y)\}_{\mathrm{PB}} = -\delta_{mn}\delta'(x-y)\,, \qquad 1 \le m,\ n \le M-k\,, \\ &\{\bar{B}_i(x), B_m(y)\}_{\mathrm{PB}} = 0 \end{aligned} \tag{6.17}$$

by imposing the constraint

$$\psi_{M+1,M-k} = \sum_{j=1}^{M+1} \bar{B}_j + \sum_{n=1}^{M-k} B_n = 0\,, \tag{6.18}$$

which is second class due to

$$\{\psi_{M+1,M-k}(x),\ \psi_{M+1,M-k}(y)\} = (k+1)\delta'(x-y)\,. \tag{6.19}$$

We are interested in describing the corresponding bracket structures associated with the generalization of the Lax operator (5.36) to

$$\mathcal{L}_{M+1,M-k} \equiv \prod_{j=1}^{M+1} (D + \bar{B}_j) \prod_{n=1}^{M-k} (D - B_n)^{-1}\,. \tag{6.20}$$

In Ref. 21 it was pointed out that the Lax operator $\mathcal{L}_{M+1,M-k}$ (with no trace condition imposed) satisfies the second Gelfand–Dickey bracket provided the fields $\bar{B}_i$, B_n satisfy (6.17). The proposition is therefore:

Proposition

$$\begin{aligned} &\{\langle\mathcal{L}_{M+1,M-k}|X\rangle, \langle\mathcal{L}_{M+1,M-k}|Y\rangle\}_{\mathrm{PB}} \\ &\quad = \mathrm{Tr}_A\left[(\mathcal{L}_{M+1,M-k}X)_+\mathcal{L}_{M+1,M-k}Y - (X\mathcal{L}_{M+1,M-k})_+Y\mathcal{L}_{M+1,M-k}\right]\,, \end{aligned} \tag{6.21}$$

where the subscript PB stands for the Poisson bracket as defined by (6.17). The statement can easily be proved by induction in M and k. It is straightforward to verify (6.21) for $M=0$, $k=0$ just by inserting $\mathcal{L}_{1,0} = D + \bar{B}$ into the formula (6.21). The essential part of the induction proof with respect to M consists in showing that (6.21) is valid for $\mathcal{L}_{M+1} = (D+\bar{B})\mathcal{L}_M$ provided it is valid for $\mathcal{L}_M$ with $k=0$. We have

$$\begin{aligned}
&\left\{\langle\mathcal{L}_{M+1}|X\rangle, \langle\mathcal{L}_{M+1}|Y\rangle\right\}_{\mathrm{PB}}\big|_{\bar{B}=\text{fixed}} \\
&\quad = \left\{\langle\mathcal{L}_M|X(D+\bar{B})\rangle, \langle\mathcal{L}_M|Y(D+\bar{B})\rangle\right\}_{\mathrm{PB}}\big|_{\mathcal{L}_M=\text{fixed}} \\
&\qquad + \left\{\langle(D+\bar{B})|\mathcal{L}_M X\rangle, \langle(D+\bar{B})|\mathcal{L}_M Y\rangle\right\}_{\mathrm{PB}} \\
&\quad = \mathrm{Tr}_A\left[\left(\mathcal{L}_M X(D+\bar{B})\right)_+ \mathcal{L}_M Y(D+\bar{B}) - \left(X(D+\bar{B})\mathcal{L}_M\right)_+ Y(D+\bar{B})\mathcal{L}_M\right] \\
&\qquad + \int dx\, \mathrm{Res}(\mathcal{L}_M X)\partial_x\, \mathrm{Res}(\mathcal{L}_M Y)\,. \qquad (6.22)
\end{aligned}$$

Now, using the simple identity for pseudodifferential operators (valid for any $\bar{B}$),

$$(D+\bar{B})\left(\mathcal{L}_M X(D+\bar{B})\right)_+ = \left((D+\bar{B})\mathcal{L}_M X\right)_+(D+\bar{B}) + \partial\, \mathrm{Res}(\mathcal{L}_M X)\,, \quad (6.23)$$

we arrive at the desired result:

$$\{\langle\mathcal{L}_{M+1}|X\rangle, \langle\mathcal{L}_{M+1}|Y\rangle\}_{\mathrm{PB}} = \mathrm{Tr}_A\left[(\mathcal{L}_{M+1}X)_+\mathcal{L}_{M+1}Y - (X\mathcal{L}_{M+1})_+ Y\mathcal{L}_{M+1}\right]. \quad (6.24)$$

The remaining step of the induction proof with respect to k, involving the transition $\mathcal{L}_{M+1,k} \to \mathcal{L}_{M+1,k+1}$, can be performed using the same techniques as above.

We now turn our attention to the Dirac bracket which results from (6.21) by imposing the constraint (6.18). The relevant statement is:

Proposition

$$\begin{aligned}
&\{\langle\mathcal{L}_{M+1,M-k}|X\rangle, \langle\mathcal{L}_{M+1,M-k}|Y\rangle\}_{\mathrm{DB}} \\
&\quad = \{\langle\mathcal{L}_{M+1,M-k}|X\rangle, \langle\mathcal{L}_{M+1,M-k}|Y\rangle\}_{\mathrm{PB}} \\
&\qquad + \frac{1}{k+1}\int dx\, \mathrm{Res}([\mathcal{L}_{M+1,M-k}, X])\partial^{-1}\, \mathrm{Res}([\mathcal{L}_{M+1,M-k}, Y])\,. \qquad (6.25)
\end{aligned}$$

The proof follows from the computation of the extra term of the relevant Dirac bracket:

$$\begin{aligned}
&-\int \{\langle\mathcal{L}_{M+1,M-k}|X\rangle, \psi_{M+1,M-k}\}_{\mathrm{PB}}\{\psi_{M+1,M-k}, \psi_{M+1,M-k}\}^{-1}_{\mathrm{DB}} \\
&\qquad \times \{\psi_{M+1,M-k}, \langle\mathcal{L}_{M+1,M-k}|Y\rangle\}_{\mathrm{DB}}\,. \qquad (6.26)
\end{aligned}$$

One easily verifies the equality of (6.26) with the second term on the r.h.s. of (6.25) using (6.19):

$$\begin{aligned}
\{\langle\mathcal{L}_{M+1,M-k}|X\rangle, \psi_{M+1,M-k}(z)\}_{\mathrm{PB}} &= -\langle[\delta(x-z), \mathcal{L}_{M+1,M-k}]|X\rangle \\
&= -\mathrm{Res}([X, \mathcal{L}_{M+1,M-k}])(z)\,. \qquad (6.27)
\end{aligned}$$

The formula (6.25) contains as special cases the KP hierarchy, corresponding to $k = 0$, and the KdV hierarchy, corresponding to $k = M$ (see e.g. Ref. 32). For the intermediary cases $0 < k < M$, Eq. (6.25) represents a compact expression for the Poisson bracket structure of all $\mathrm{SL}(M+1,\ M-k)$ KdV hierarchies defined by the Lax operators (6.20).

6.2. *The Lax formulation of* $\mathrm{SL}(M+1, M-k)$ *KdV*

In this subsection we give expressions for the coefficients of the Lax operators of the generalized $\mathrm{SL}(M+1,\ M-k)$ KdV hierarchy in terms of free fields. Let us start with the case where all the currents B_i are gauged away according to (6.14). In this limit the expression (5.26) becomes

$$\tilde{A}_M \equiv \tilde{A}_M^{(M)} = \sum_{k=1}^{M} (\partial + \tilde{c}_k) \left(- \sum_{l=k}^{M} \tilde{c}_l \right) , \tag{6.28}$$

$$\begin{aligned} \tilde{A}_k \equiv \tilde{A}_k^{(M)} = & \sum_{n_{M-k+1}=1}^{k} (\partial + \tilde{c}_{n_{M-k+1}+M-k}) \\ & \times \sum_{n_{M-k}=1}^{n_{M-k+1}} (\partial + \tilde{c}_{n_{M-k}+M-1-k}) \times \cdots \times \sum_{n_2=1}^{n_3} (\partial + \tilde{c}_{n_2+1}) \\ & \times \sum_{n_1=1}^{n_2} (\partial + \tilde{c}_{n_1}) \left(- \sum_{l=n_1}^{M} \tilde{c}_l \right) , \qquad k = 1, \dots , M-1 , \end{aligned} \tag{6.29}$$

which agrees with the Fateev–Lukyanov[22] expression:

$$\begin{gathered} \prod_{i=1}^{M+1} (D + \bar{B}_i) = D^{M+1} + \tilde{A}_M D^{M-1} + \cdots + \tilde{A}_1 , \\ \bar{B}_{M+1} = - \sum_{l=1}^{M} \tilde{c}_l , \qquad \bar{B}_i = \tilde{c}_{M+1-i} , \qquad i = 1, \dots , M . \end{gathered} \tag{6.30}$$

The reason for this agreement is that the Lax operator in (6.30) is equal to $\mathcal{L}_{M+1,0}$ [Eq. (6.20) for $k = 0$] with the condition $\psi_{M+1,0} = 0$ imposed. Both approaches (Dirac bracket and gauge-fixing) lead to the algebra (6.15) or, equivalently, to the formula (6.25) with $k = M$. Hence, our construction has provided a simple proof for the Fateev–Lukyanov expression[22] (see also Refs. 6, 33). Note that the use of the gauge-fixing method has the advantage that the Lax coefficients (6.29) of the $\mathrm{SL}(M+1)$ KdV can still be written in terms of the original Darboux–Poisson pairs $(c_r, e_r)_{r=1}^{M}$ through $\tilde{c}_i$ from (6.14), which is given by

$$\tilde{c}_i = -e_i - \sum_{l=i+1}^{M} c_l + \frac{1}{M+1} \sum_{l=1}^{M} (e_l + l c_l) , \qquad i = 1, \dots , M . \tag{6.31}$$

Generally, to obtain a convenient expression for the Lax operator of the SL($M+1$, $M-k$) KdV associated with the graded SL($M+1$, $M-k$) algebra, we gauge away $B_M, \dots, B_{M-k+1}$ leaving $\tilde{B}_1, \dots, \tilde{B}_{M-k}$. We obtain in this way from (5.34)

$$\mathcal{L}_{M+1,M-k} = \prod_{l=M}^{M-k+1} (D+\tilde{c}_l) \prod_{l=M-k}^{1} (D+\tilde{c}_l - \tilde{B}_l) \times \left(D - \sum_{l=1}^{M} \tilde{c}_l\right) \prod_{l=1}^{M-k} (D-\tilde{B}_l)^{-1} , \tag{6.32}$$

which automatically satisfies the appropriate trace condition (6.18). We can interpret (6.32) as a superdeterminant of the graded SL($M+1$, $M-k$) matrix in a diagonal gauge, which for the ordinary KdV case $k=M$ becomes an ordinary determinant as in (6.30).

We can alternatively rewrite the last expression (6.32) in a way which corresponds to the DS gauge as

$$\mathcal{L}_{M+1,M-k} = \sum_{l=1}^{M-k} \tilde{A}_l \prod_{i=l}^{M-k} (D-\tilde{B}_i)^{-1} + \sum_{l=0}^{k-1} \tilde{A}_{l+M-k+1} D^l + D^{k+1} , \tag{6.33}$$

with the second bracket structure automatically given by the formula (6.25). The coefficients $\tilde{A}_l$ can be explicitly expressed in terms of $\tilde{c}_l$, $\tilde{B}_l$ from (5.25) and (5.26) by substituting there

$$\begin{aligned} c_l \to \tilde{c}_l , \qquad e_l \to \tilde{B}_l - \sum_{i=l}^{M} \tilde{c}_i , \qquad l = 1, \dots, M-k , \\ e_l \to -\sum_{i=l}^{M} \tilde{c}_i , \qquad l = M-k+1, \dots, M . \end{aligned} \tag{6.34}$$

Hence, again, we arrived at representation of the coefficients of the SL($M+1$, $M-k$) KdV Lax operators (6.33), or (6.32), in terms of the free fields (currents) whose Poisson bracket algebra is given by (6.10) (recall that $\bar{B}_i = \tilde{c}_{M-i+1}$ for $1 \le i \le M-k$ and $\bar{B}_i = \tilde{c}_{M-i+1} - \tilde{B}_{M-i+1}$ for $M-k+1 \le i \le M$). Correspondingly, the Lax coefficients $\tilde{A}_i$, $\tilde{B}_j$ in (6.33) satisfy a nonlinear Poisson bracket algebra, called $W(M, M-k)$ algebra in Ref. 18, which results from (6.25). This $W(M, M-k)$ algebra is a generalization of the well-known Zamolodchikov nonlinear W_{M+1} algebra,[25] in particular $W(M,0) \simeq W_{M+1}$. Thus, Eqs. (5.25) and (5.26) with the substitutions (6.34) provide explicit free field realization of $W(M, M-k)$.

The Lax operator $\mathcal{L}_{M+1,M-k}$ [Eq. (6.32)] [or Eq. (6.33)] possesses the following pseudodifferential series expansion:

$$\mathcal{L}_{M+1,M-k} = D^{k+1} + \sum_{l=0}^{k-1} \tilde{A}_{l+M-k+1}(\tilde{c},\tilde{B})D^l + \sum_{n=0}^{\infty} w_n(\tilde{c},\tilde{B})D^{-1-n}\,, \tag{6.35}$$

$$w_n(\tilde{c},\tilde{B}) = \sum_{r=0}^{\min(M-k-1,n)} (-1)^{n-r}\tilde{A}_{M-k-r}(\tilde{c},\tilde{B})P_{n-r}^{(r+1)} \times \left(-\tilde{B}_{M-k-r},\dots,-\tilde{B}_{M-k}\right). \tag{6.36}$$

Here, as above, $\tilde{A}_s(\tilde{c},\tilde{B})$ are given by the expressions (5.25) and (5.26) with the substitutions (6.34), whereas

$$P_n^{(r)}(\phi_1,\dots,\phi_r) \equiv \sum_{m_1+\cdots+m_r=n} (\partial+\phi_1)^{m_1}\cdots(\partial+\phi_r)^{m_r}\cdot 1 \tag{6.37}$$

denotes the multiple Faá di Bruno polynomials [cf. Refs. 8 and 20 for analogous-to-(6.35)–(6.37) expressions for the multiboson KP Lax operators]. The Poisson bracket algebra of the coefficient fields $\tilde{A}_s(\tilde{c},\tilde{B})$, $s = M-k+1,\dots,M$, and $w_n(\tilde{c},\tilde{B})$, $n=0,1,2,\dots$, which results from the free field Poisson brackets of their constituents (6.10), is a nonlinear algebra $\hat{\mathbf{W}}_\infty^{(\mathbf{k})}$ generalizing the known nonlinear $\hat{\mathbf{W}}_\infty$ algebra[34] (see also Ref. 21). In particular, $\hat{\mathbf{W}}_\infty^{(\mathbf{k=0})} \simeq \hat{\mathbf{W}}_\infty$.

Example — SL(3,1) KdV hierarchy

This is defined by the Lax operator $\mathcal{L}_{3,1} = \tilde{A}_1(\partial-\tilde{B}_1)^{-1} + \tilde{A}_2 + \partial^2$ [cf. (6.33)], where

$$\tilde{A}_1 = (\partial+\tilde{B}_1+\tilde{c}_2)(\partial+\tilde{c}_1)(\tilde{B}_1-\tilde{c}_1-\tilde{c}_2)\,, \tag{6.38}$$

$$\tilde{A}_2 = (\partial+\tilde{c}_1)(\tilde{B}_1-\tilde{c}_1) + (\tilde{c}_1-\tilde{c}_2)\tilde{c}_2\,, \tag{6.39}$$

and with fundamental Poisson brackets:

$$\begin{aligned} &\{\tilde{c}_1(x),\tilde{B}_1(y)\} = -\delta'(x-y)\,, && \{\tilde{c}_2(x),\tilde{c}_2(y)\} = \frac{1}{2}\delta'(x-y)\,,\\ &\{\tilde{c}_2(x),\tilde{B}_1(y)\} = -\frac{1}{2}\delta'(x-y)\,, && \{\tilde{B}_1(x),\tilde{B}_1(y)\} = -\frac{3}{2}\delta'(x-y)\,. \end{aligned} \tag{6.40}$$

The brackets (6.40) imply that $\tilde{A}_{1,2}$ given by (6.38), and (6.39), together with $\tilde{B}_1$, satisfy the $W(2,1)$ Poisson bracket algebra (4.7) (with $\mathcal{B} \equiv \tilde{B}_1$).

Now, one notices the presence of the zero order term $\tilde{A}_{M-k+1}$ in the Lax operator $\mathcal{L}_{M+1,M-k}$ [Eq. (6.33)]. This fact enables us to prove that the SL$(M+1,M-k)$ KdV hierarchy is a bi-Hamiltonian hierarchy. Consider, namely, $\mathcal{L}'_{M+1,M-k} = \mathcal{L}_{M+1,M-k} - \lambda$, obtained by redefining the zero order term in the Lax operator by addition of the constant λ. Clearly the second bracket (6.25) for the new Lax operator becomes

$$\begin{aligned}
&\left\{\langle\mathcal{L}'_{M+1,k}|X\rangle, \langle\mathcal{L}'_{M+1,k}|Y\rangle\right\}_{\rm DB} \\
&\quad = \mathrm{Tr}_A\left[(\mathcal{L}'_{M+1,k}X)_+\mathcal{L}'_{M+1,k}Y - (X\mathcal{L}'_{M+1,k})_+Y\mathcal{L}'_{M+1,k}\right] \\
&\quad + \frac{1}{k+1}\int dx\, \mathrm{Res}\big([\mathcal{L}'_{M+1,k}, X]\big)\partial^{-1}\,\mathrm{Res}\big([\mathcal{L}'_{M+1,k}, Y]\big) \\
&\quad - \lambda\langle\mathcal{L}'_{M+1,k}|[X,Y]_R\rangle\,,
\end{aligned} \tag{6.41}$$

where we have introduced the R commutator $[X,Y]_R \equiv [X_+,Y_+] - [X_-,Y_-]$. Here again the subscripts $\pm$ denote projections on the pure differential and pseudodifferential parts of the pseudodifferential operators X, Y, respectively. Define next an R bracket $\{\cdot\,,\cdot\}_1^R$ as a bracket obtained from (1.1) by substituting the R commutator $[X,Y]_R$ for the ordinary commutator:[19,35]

$$\{\langle L|X\rangle, \langle L|Y\rangle\}_1^R \equiv -\langle L|[X,Y]_R\rangle\,. \tag{6.42}$$

The relation (6.41) thus shows that the linear combination of brackets $\{\cdot\,,\cdot\}_{\rm DB} + \lambda\{\cdot\,,\cdot\}_1^R$ satisfies the Jacobi identity. We can state this result as:

Proposition. The $\mathrm{SL}(M+1,\ M-k)$ *KdV hierarchy is bi-Hamiltonian with brackets* $\{\cdot\,,\cdot\}_{\rm DB}$ *and* $\{\cdot\,,\cdot\}_1^R$ *defining a compatible pair of Hamiltonian structures.*

This proposition establishes, therefore, the fundamental criterion for integrability of the generalized $\mathrm{SL}(M+1,\ M-k)$ KdV hierarchy.

6.3. *The discrete symmetry of* $\mathrm{SL}(M+1, M-k)$ *KdV hierarchy*

Recently, the multiboson KP hierarchies have been shown to possess canonical discrete symmetry realized as a similarity transformation of their Lax operator.[14] It is natural to ask whether the discrete similarity transformation can be constructed for the reduced $\mathrm{SL}(M+1,\ M-k)$ KdV hierarchy for $k \neq 0$. One suspects that the presence of B currents in this reduction will allow remnants of the discrete symmetry to survive in this system. We shall show now that this is indeed the case.

First, let us consider the simplest nontrivial example — the pseudodifferential Lax operator of the SL(3,1) KdV hierarchy (for convenience here we suppress the tildes on the coefficients fields):

$$\mathcal{L}_{3,1} = A_1\,\frac{1}{\partial - B_1} + A_2 + \partial^2\,. \tag{6.43}$$

It is easy to prove its covariance under the similarity transformation:

$$(\partial - \mathcal{B}^0)\mathcal{L}_{3,1}(\partial - \mathcal{B}^0)^{-1} = \bar{A}_1\,\frac{1}{\partial - \mathcal{B}^0} + \bar{A}_2 + \partial^2\,, \tag{6.44}$$

provided $\mathcal{B}^0 = B_1 + \partial \ln A_1$. Equation (6.44) induces the following discrete transformations on the Lax coefficients which can be viewed as auto-Bäcklund transformations for the underlying SL(3,1) KdV hierarchy:

$$
\begin{aligned}
B_1 \to \bar{B}_1 &= \mathcal{B}^0 = \mathcal{B} + \partial \ln A_1 \,, \\
A_2 \to \bar{A}_2 &= A_2 + 2\partial(\mathcal{B} + \partial \ln A_1) \,, \\
A_1 \to \bar{A}_1 &= A_1 + A_2' + \partial\left[(\mathcal{B} + \partial \ln A_1)^2 + \partial(\mathcal{B} + \partial \ln A_1)\right] .
\end{aligned} \tag{6.45}
$$

This can be represented by the following Toda-like lattice equations of motion:

$$
\begin{aligned}
\partial a_2(n) &= a_2(n)[a_0(n+1) - a_0(n)] \,, \\
\partial a_0(n+1) &= \frac{1}{2}\left[a_1(n+1) - a_1(n)\right] , \\
\partial a_1(n) &= a_2(n+1) - a_2(n) - \partial[a_0^2(n+1) + \partial a_0(n+1)] \,,
\end{aligned} \tag{6.46}
$$

upon identifying

$$
\begin{aligned}
a_2(n) &\simeq A_1 \,, & a_1(n) &\simeq A_2 \,, & a_0(n) &\simeq B_1 \,, \\
a_2(n+1) &\simeq \bar{A}_1 \,, & a_1(n+1) &\simeq \bar{A}_2 \,, & a_0(n+1) &\simeq \bar{B}_1 \,.
\end{aligned} \tag{6.47}
$$

Equations (6.46) can be obtained as consistency conditions of the following lattice spectral system:

$$
\begin{aligned}
a_2(n)\Psi_{n-1} + a_1(n)\Psi_n + \partial^2\Psi_n &= \lambda\Psi_n \,, \\
\partial\Psi_{n-1} - a_0(n)\Psi_{n-1} &= \Psi_n \,.
\end{aligned} \tag{6.48}
$$

The above discrete symmetry extends to the general case given by the Lax operator (6.33). We find that the similarity transformation $(\partial - \mathcal{B}^0)\mathcal{L}_{M+1,M-k}$ $(\partial - \mathcal{B}^0)^{-1}$ with $\mathcal{B}^0 = B_1 + \partial \ln A_1$ again preserves the form of the Lax operator, while its coefficients undergo the transformations

$$
\begin{aligned}
B_l \to \bar{B}_l &= B_{l+1} \,, \\
A_l \to \bar{A}_l &= A_l + (\partial + B_{l+1} - \mathcal{B}^0)A_{l+1}
\end{aligned} \tag{6.49}
$$

for the first $M-k-1$ coefficients labeled by $l = 1, \dots, M-k-1$ and behaving under similarity transformation in a way consistent with the lattice site translations in the underlying Toda-like lattice[14] [cf. Eqs. (6.46)]. For the remaining coefficients we find that

$$
\begin{aligned}
B_{M-k} \to \bar{B}_{M-k} &= \mathcal{B}^0 = B_1 + \partial \ln A_1 \,, \\
A_{M-k} \to \bar{A}_{M-k} &= A_{M-k} + P_{k+1}'(\mathcal{B}^0) + \sum_{l=0}^{k-1} \left(A_{M-k+l+1} P_l(\mathcal{B}^0)\right)' ,
\end{aligned}
$$

$$A_{M-k+\alpha} \to \bar{A}_{M-k+\alpha} = \sum_{m=1}^{m}\sum_{p=0}^{m-2}\binom{m-1}{p}\binom{m-2-p}{\alpha-1}$$
$$\times \left(A_{M-k+m}P_p(\mathcal{B}^0)\right)' P_{m-1-p-\alpha}(-\mathcal{B}^0)$$
$$+\sum_{m=1}^{k}\sum_{p=0}^{m-1}\binom{m-1}{p}\binom{m-1-p}{\alpha-1}$$
$$\times A_{M-k+m}P_p(\mathcal{B}^0)P_{m-p-\alpha}(-\mathcal{B}^0)$$
$$+\sum_{p=1}^{k}\binom{k+1}{p}\binom{k-p}{\alpha-1}P_p'(\mathcal{B}^0)P_{k+1-p-\alpha}(-\mathcal{B}^0)$$
$$+\sum_{p=0}^{k+1}\binom{k+1}{p}\binom{k-p+1}{\alpha-1}P_p(\mathcal{B}^0)P_{k+2-p-\alpha}(-\mathcal{B}^0)\,, \tag{6.50}$$

for $1 \leq \alpha \leq k$. The symbols $P_n(\pm\phi) \equiv (D \pm \phi)^n \cdot 1$ denote the ordinary Faá di Bruno polynomials [cf. (6.37)]. Equations (6.49) and (6.50) represent the auto-Bäcklund transformations for the generalized SL($M+1$, $M-k$) KdV hierarchies. Both the Hamiltonians and Poisson structures of the underlying hierarchies are invariant under (6.49) and (6.50) due to the similarity character of these auto-Bäcklund transformations.

7. Two-Matrix Model as an SL($M+1, 1$) KdV Hierarchy

Now, let us show how the formalism of the previous sections finds application in the two-matrix string model. In what follows we present a generalization and extension of the discussion from Ref. 24.

$$Z_n[t,\tilde{t},g] = \int dM_1\, dM_2\, \exp - \left\{\sum_{r=1}^{p_1} t_r \;\mathrm{Tr}\; M_1^r + \sum_{s=1}^{p_2} \tilde{t}_s \;\mathrm{Tr}\; M_2^s + g\;\mathrm{Tr}\; M_1M_2\right\}. \tag{7.1}$$

Here $M_{1,2}$ are Hermitian $N \times N$ matrices, and the orders of the matrix "potentials" $p_{1,2}$ are assumed to be finite. In Refs. 11 and 23 it was shown that, by the method of generalized orthogonal polynomials,[36] the partition function (7.1) and its derivatives w.r.t. the parameters $(t_r, \tilde{t}_s, g)$ can be explicitly expressed in terms of solutions to constrained generalized Toda lattice hierarchies associated with (7.1). The corresponding linear problem and Lax (or "zero curvature") representation of the latter read[11,23]

$$Q_{nm}\psi_m = \lambda\psi_n\,, \qquad -g\bar{Q}_{nm}\psi_m = \frac{\partial}{\partial\lambda}\psi_n\,, \tag{7.2}$$

$$\frac{\partial}{\partial t_r}\psi_n = \left(Q^r_{(+)}\right)_{nm}\psi_m\,, \qquad \frac{\partial}{\partial \tilde{t}_s}\psi_n = -\left(\bar{Q}^s_{-}\right)_{nm}\psi_m\,, \tag{7.3}$$

$$\frac{\partial}{\partial \tilde{t}_s}\bar{Q} = [\bar{Q}, \bar{Q}^s_{-}]\,, \qquad \frac{\partial}{\partial \tilde{t}_s}Q = [Q, \bar{Q}^s_{-}]\,, \tag{7.4}$$

$$\frac{\partial}{\partial t_r}Q = \left[Q^r_{(+)}, Q\right]\,, \qquad \frac{\partial}{\partial t_r}\bar{Q} = \left[Q^r_{(+)}, \bar{Q}\right]\,, \tag{7.5}$$

$$-g[Q,\bar{Q}] = \mathbb{1}\,. \tag{7.6}$$

The subscripts $-/+$ denote lower/upper-triangular parts, whereas $(+)/(-)$ denote upper/lower-triangular plus diagonal parts. The parametrization of the matrices Q and $\bar{Q}$ is as follows:

$$\begin{aligned} Q_{nm} &= a_0(n)\,, \qquad Q_{n,n+1} = 1\,, \\ Q_{n,n-k} &= a_k(n)\,, \qquad k = 1,\dots,p_2-1\,, \\ Q_{nm} &= 0 \quad \text{for} \quad m-n \geq 2\,, \qquad n-m \geq p_2\,, \end{aligned} \tag{7.7}$$

$$\begin{aligned} \bar{Q}_{nn} &= b_0(n)\,, \qquad \bar{Q}_{n,n-1} = R_n\,, \\ \bar{Q}_{n,n+k} &= b_k(n)R^{-1}_{n+1}\cdots R^{-1}_{n+k}\,, \qquad k = 1,\dots,p_1-1\,, \\ \bar{Q}_{nm} &= 0 \quad \text{for} \quad n-m \geq 2\,, \qquad m-n \geq p_1\,. \end{aligned} \tag{7.8}$$

Let us also introduce special notations for the first evolution parameters t_1, $\tilde{t}_1$, which in the sequel will be considered as space coordinates, i.e. $\tilde{t}_1 \equiv x$ and $t_1 \equiv y$. The lattice equations of motion (7.4) with $s = 1$ imply that:

(a) All matrix elements of $\bar{Q}$ can be expressed as functionals (w.r.t. x) of R_{n+1}, $b_0(n), \dots, b_{p_1-2}(n)$ at a *fixed* lattice site n;
(b) All matrix elements of Q are explicitly expressed through R_{n+1}, $b_0(n), \dots, b_{p_1-2}(n)$ via the formula[11]

$$Q_{(-)} = \sum_{s=0}^{p_2-1} \alpha_s \bar{Q}^s_{(-)}\,, \qquad \alpha_s \equiv -(s+1)\,\frac{\tilde{t}_{s+1}}{g}\,, \tag{7.9}$$

where the coefficients α_s are determined from matching with the "string" equation (7.6).

There is a complete duality under the interchange $p_1 \leftrightarrow p_2$ of the orders of the matrix potentials in (7.1), supplemented by interchanging $\{t_r\} \leftrightarrow \{\tilde{t}_s\}$, $Q_{(-)} \leftrightarrow \bar{Q}_{(+)}$.[24]

Using the parametrization (7.7)–(7.8), the equations of the auxiliary linear Lax problem (7.2), (7.3) acquire the form

$$\lambda\psi_n = \psi_{n+1} + a_0(n)\psi_n + \sum_{k=1}^{p_2-1} a_k(n)\psi_{n-k}\,, \tag{7.10}$$

$$-\frac{1}{g}\frac{\partial}{\partial\lambda}\psi_n = R_n\psi_{n-1} + b_0(n)\psi_n + \sum_{k=1}^{p_1-1} \frac{b_k(n)}{R_{n+1}\cdots R_{n+k}}\psi_{n+k}\,, \tag{7.11}$$

$$\partial_x\psi_n = -R_n\psi_{n-1}\,, \qquad \partial_y\psi_n = \psi_{n+1} + a_0(n)\psi_n\,. \tag{7.12}$$

Comparing Eqs. (7.10)–(7.12) with (2.1), one identifies the two-matrix model as a special constrained Toda lattice hierarchy. Using (7.12), the implications of the lattice equations of motion:

$$\begin{aligned}
a_0(n+k) &= a_0(n) + \partial_y \ln(R_{n+1}\cdots R_{n+k-1})\,,\\
a_0(n-k) &= a_0(n) - \partial_y \ln(R_n\cdots R_{n-k+1})\,,\\
b_0(n+k) &= b_0(n) + \partial_x \ln(R_{n+1}\cdots R_{n+k-1})\,,\\
b_0(n-k) &= b_0(n) - \partial_x \ln(R_n\cdots R_{n-k+1})\,,
\end{aligned}$$

and the string equation solution (7.9), one can rewrite (7.10)–(7.12) and their compatibility conditions as a continuum Lax problem at a fixed lattice site n:

$$\begin{aligned}
\lambda\psi_n &= L(n)\psi_n\,, & -\frac{1}{g}\frac{\partial}{\partial\lambda}\psi_n &= \bar{L}(n)\psi_n\,,\\
\frac{\partial}{\partial t_r}\psi_n &= \mathcal{L}_r(n)\psi_n\,, & \frac{\partial}{\partial \tilde{t}_s}\psi_n &= -\bar{\mathcal{L}}_s(n)\psi_n\,,
\end{aligned} \tag{7.13}$$

$$\frac{\partial}{\partial t_r}L(n) = [\mathcal{L}_r(n), L(n)]\,, \qquad \frac{\partial}{\partial \tilde{t}_s}L(n) = [L(n), \bar{\mathcal{L}}_s(n)]\,, \tag{7.14}$$

$$\frac{\partial}{\partial t_r}\bar{L}(n) = [\mathcal{L}_r(n), \bar{L}(n)]\,, \qquad \frac{\partial}{\partial \tilde{t}_s}\bar{L}(n) = [\bar{L}(n), \bar{\mathcal{L}}_s(n)]\,, \tag{7.15}$$

$$[L(n), \bar{L}(n)] = -\frac{1}{g}\,\mathbb{1}\,. \tag{7.16}$$

The above form of the continuum Lax equations is not convenient since the coefficients of the continuum Lax operators depend explicitly on the evolution parameters $\tilde{t}_s$ through the coefficients α_s in the string equation solution (7.9). Here we shall perform a suitable transformation of both the independent functions R_{n+1}, $b_k(n) \to \hat{R}_{n+1}$, $\hat{b}_k$, and the subset of evolution parameters $\{\tilde{t}_s\} \to \{\hat{t}_s\}$. The result will be the unification of all of Eqs. (7.14)–(7.16) into a single set of flow equations where the Lax operator coefficients will not contain any explicit dependence on the new evolution parameters.

Let us introduce another matrix $\hat{Q}$ [with matrix elements $\hat{R}_n$, $\hat{b}_k(n)$; cf. (7.8)] in place of $\bar{Q}$ defined as follows (recall $\mathcal{E}_{nm} = \delta_{n+1,m}$):

$$\hat{Q}_{(-)}^{p_2-1} = \sum_{s=0}^{p_2-1} \alpha_s \bar{Q}_{(-)}^s , \qquad \hat{Q}_{+}^{p_2-1} = \mathcal{E} \to \hat{Q}^{p_2-1} = Q , \tag{7.17}$$

where the last equality follows from (7.9). The structure of the matrices $\hat{Q}$ and $\bar{Q}$ imply that in fact the more general system of equations is satisfied:

$$\hat{Q}_{(-)}^s = \sum_{\sigma=0}^{s} \gamma_{s\sigma} \bar{Q}_{(-)}^{a} , \qquad s = 0, 1, \dots , p_2 , \tag{7.18}$$

with coefficients $\gamma_{s\sigma}$ simply expressed through $\alpha_s \equiv \gamma_{p_2-1,s}$ (7.9) ($\gamma_{00} \equiv 1$). Explicitly we get

$$\begin{aligned} \gamma_{ss} &= (\gamma_{11})^s , \\ \gamma_{s,s-1} &= \frac{s}{2} (\gamma_{11})^{s-2} \gamma_{21} , \\ \gamma_{s,s-2} &= (\gamma_{11})^{s-4} \left[\frac{s(s-3)}{8} (\gamma_{21})^3 + \frac{s}{3} \gamma_{11} \gamma_{31} \right] , \end{aligned} \tag{7.19}$$

$$\begin{aligned} \gamma_{11} &\equiv (\alpha_{p_2-1})^{\frac{1}{p_2-1}} , \\ \gamma_{21} &\equiv \frac{2}{p_2-1} \frac{\alpha_{p_2-2}}{(\alpha_{p_2-1})^{\frac{p_2-3}{p_2-1}}} , \\ \gamma_{31} &\equiv \frac{3}{p_2-1} \left[\frac{\alpha_{p_2-3}}{(\alpha_{p_2-1})^{\frac{p_2-4}{p_2-1}}} - \frac{p_2-4}{2(p_2-1)} \frac{\alpha_{p_2-2}^2}{(\alpha_{p_2-1})^{\frac{2p_2-5}{p_2-1}}} \right] , \end{aligned} \tag{7.20}$$

and for the $\hat{Q}$-matrix elements we obtain

$$\begin{aligned} \hat{R}_n &= \gamma_{11} R_n , \\ \hat{b}_0(n) &= \gamma_{11} b_0(n) + \frac{\gamma_{21}}{2\gamma_{11}} , \\ \hat{b}_1(n) &= \gamma_{11}^2 b_1(n) + \frac{\gamma_{31}}{3\gamma_{11}} - \left(\frac{\gamma_{21}}{2\gamma_{11}} \right)^2 , \end{aligned} \tag{7.21}$$

etc. Let us also introduce the new subset of evolution parameters $\{\tilde{t}_s\}$ through the equations

$$\frac{\partial}{\partial \hat{t}_s} = \sum_{\sigma=1}^{s} \gamma_{s\sigma} \frac{\partial}{\partial \tilde{t}_\sigma} , \qquad s = 1, \dots , p_2 . \tag{7.22}$$

Now, taking into account (7.17) and (7.22), all constrained Toda lattice equations (7.4)–(7.6) can be re-expressed as a single set of flow equations for the (unconstrained) matrix $\hat{Q}$:

$$\frac{\partial}{\partial \hat{t}_s} \hat{Q} = [\hat{Q}, \hat{Q}^s_-] , \quad s = 1, \dots , p_2 , \quad 2(p_2 - 1) , \quad 3(p_2 - 1), \dots , p_1(p_2 - 1) , \tag{7.23}$$

with the identification $t_r \equiv \hat{t}_{r(p_2-1)}$, whose associated linear problem reads

$$\hat{Q}_{nm} \hat{\psi}_m = \lambda \hat{\psi}_n , \qquad \frac{\partial}{\partial \hat{t}_s} \hat{\psi}_n = -(\hat{Q}^s_-)_{nm} \hat{\psi}_m . \tag{7.24}$$

Using the same procedure as in the derivation of Eqs. (7.13)–(7.16) from (7.10)–(7.12), we obtain from (7.23) and (7.24) the following continuum Lax problem at a fixed lattice site n:

$$\lambda^{p_2-1} \hat{\psi}_n = \hat{L}(n) \hat{\psi}_n , \qquad \frac{\partial}{\partial \hat{t}_s} \hat{\psi}_n = -\hat{\mathcal{L}}_s(n) \hat{\psi}_n , \tag{7.25}$$

$$\frac{\partial}{\partial \hat{t}_s} \hat{L}(n) = [\hat{L}(n), \hat{\mathcal{L}}_s(n)] , \quad s = 1, \dots , p_2 , \quad 2(p_2 - 1) , \quad 3(p_2 - 1), \dots , p_1(p_2 - 1) , \tag{7.26}$$

$$\begin{aligned} \hat{L}(n) \equiv & - D_x^{-1} \hat{R}_{n+1} + \hat{Q}^{p_2-1}_{nm} + \sum_{k=1}^{p_2-1} \frac{(-1)^k \hat{Q}^{p_2-1}_{n,n-k}}{\hat{R}_n \cdots \hat{R}_{n-k+1}} \\ & \times \left(D_x - \partial_x \ln(\hat{R}_n \cdots \hat{R}_{n-k+2})\right) \times \cdots \times (D_x - \partial_x \ln \hat{R}_n) D_x , \end{aligned} \tag{7.27}$$

$$\hat{\mathcal{L}}_s(n) \equiv \sum_{k=1}^{s} \frac{(-1)^k \hat{Q}^s_{n,n-k}}{\hat{R}_n \cdots \hat{R}_{n-k+1}} \left(D_x - \partial_x \ln(\hat{R}_n \cdots \hat{R}_{n-k+2})\right) \cdots D_x , \tag{7.28}$$

where all coefficients are expressed in terms of $\hat{R}_{n+1}$, $\hat{b}_0(n), \dots , \hat{b}_{p_1-2}(n)$ at a fixed site n through the $\hat{t}_1 \equiv x$ lattice equations of motion (7.23). Up to gauge transformation and conjugation, the explicit form of $\hat{L}(n)$ reads

$$\begin{aligned} & e^{\int \hat{b}_0(n)} \big(\hat{L}(n)\big)^* e^{-\int \hat{b}_0(n)} \\ & \quad = D_x^{p_2-1} + (p_2 - 1)\hat{b}_1(n) D_x^{p_2-3} + \cdots + \hat{R}_{n+1} \big(D_x - \hat{b}_0(n)\big)^{-1} . \end{aligned} \tag{7.29}$$

Equations (7.25)–(7.28) are the continuum analogs of the constrained Toda lattice Lax equations (7.2)–(7.6) without taking any continuum (double-scaling) limit. Let us particularly stress that they explicitly incorporate the whole information from the matrix-model string equation (7.6) through (7.9) and (7.17)–(7.21), which were used in their derivation. Thus, according to (7.26) and (7.29) the constrained generalized

Toda lattice hierarchy (7.2)–(7.6), describing the two-matrix string model (7.1), is equivalent to the SL(p_2, 1)-KdV hierarchy [cf. (6.33)] with a finite number of flows (when p_1 is finite).

8. Conclusions

We conclude with a list of remarks:

- We have described here the process of reduction of the multiboson KP hierarchies analyzed in two special settings, or gauges, of the associated Toda matrix spectral problem. Our basic gauge was the Drinfeld–Sokolov gauge naturally associated with the Toda lattice hierarchy. Another gauge, which proved to be useful in our discussion, especially when dealing with the Poisson bracket structure, was the diagonal gauge related to the Volterra lattice. Modes associated with the Volterra lattice Abelianized and simplified the analysis of the second Hamiltonian structure. These two gauges led to two different Lax formulations having different transformation properties under the residual gauge transformations.
- We have obtained in this paper a coherent approach to describe generalized KP–KdV [i.e. SL($M+1$, $M-k$) KdV] hierarchies and their Poisson bracket structures. Variables (free currents) Abelianizing the second bracket structure were instrumental in this analysis. They naturally lead to the appearance of the graded SL($M+1$, $M-k$) Kac–Moody algebras. This raises the question about the origin of the graded algebra in this setting. One possible explanation could be given in terms of the underlying lattice structure when one recalls that the transition from Toda to Volterra lattice involves separation at even and odd sites. This could be an intuitive way of seeing how the grading could have been introduced into the formalism.
- The free fields (currents) Abelianizing the second Hamiltonian structure of the KP–KdV hierarchies bring about another noticeable result. Namely, they yield explicit free field representations of the nonlinear $W(M, M-k)$ algebras, isomorphic to the SL($M+1$, $M-k$) KdV Poisson bracket algebras, which generalize Zamolodchikov's nonlinear W_{M+1} algebra.
- Lattice translations in the underlying Toda and Toda-like hierarchies naturally give rise to similarity transformations of the corresponding KP–KdV Lax operators which, first, preserve the Hamiltonians and the Poisson structures and, second, systematically generate the pertinent auto-Bäcklund transformations for the generalized SL($M+1$, $M-k$) KdV hierarchies.
- The physical relevance of the structures, defined by the reduction process described in this paper, is now strongly enhanced by the natural appearance of the generalized SL($M+1$, 1) KdV hierarchy within the context of the two-matrix string model. More precisely, Ref. 24 identified the two-matrix model in the simplest nontrivial case with a coupled system of (2 + 1)-dimensional KP and modified KP [$(m)\mathrm{KP}_{2+1}$] integrable equations subject to a specific "symmetry" constraint. This constraint together with the Miura–Konopelchenko map[37] for

$(m)\mathrm{KP}_{2+1}$ are the images in the continuum of the matrix model string equation (7.6). In particular, the two-matrix model susceptibility $\partial_{\tilde{t}_1}^2 \ln Z_N = b_1(N-1)$ is simply related via (7.21) and (7.20) with $\hat{b}_1(N-1)$, which is a solution to the above string-constrained KP_{2+1} equation. The string-constrained $(m)\mathrm{KP}_{2+1}$ system was shown to be equivalent to the $(1+1)$-dimensional generalized SL(3, 1)-KdV hierarchy (7.29) (with $p_2 = 3$), (6.43). This result was generalized here for arbitrary order potentials of the two-matrix model.[a] Furthermore, according to Eq. (7.29) $\hat{b}_1(n) = \mathrm{Res}\big(\hat{L}(n)\big)^{\frac{1}{p_2-1}} = \partial_x^2 \ln \tau_{\mathrm{SL}(p_2,1)}$ and, as shown in Ref. 38, one obtains explicit Wronskian solutions for the SL(p,q)-KdV τ-functions via special Darboux–Bäcklund transformations preserving the constrained SL(p,q)-KdV form of the pertinent Lax operators.

Acknowledgments

A. H. Z. thanks UIC for hospitality during his stay in Chicago and FAPESP for financial support. H. A. thanks IFT-UNESP for hospitality during the completion of this paper and FAPESP for financial support. E. N. and S. P. gratefully acknowledge support from the Ben-Gurion University, Beer-Sheva. Also, they express their gratitude to Prof. K. Pohlmeyer for cordial hospitality at the University of Freiburg, where part of this work was done. S. P. appreciates financial support by the *Deutscher Akademischer Austauschdienst* for her visit at the University of Freiburg. The authors are grateful to J. F. Gomes for discussions.

References

1. L. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer, 1987); A. Das, *Integrable Models* (World Scientific, 1989); M. Ablowitz and P. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge, New York, 1991); V. Korepin, N. Bogoliubov and A. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, 1993).
2. A. Polyakov, in *Fields, Strings and Critical Phenomena*, eds. E. Brézin and J. Zinn-Justin (North-Holland, 1989).
3. D. Gross and A. Migdal, *Nucl. Phys.* **B340**, 333 (1990); E. Brézin and V. Kazakov, *Phys. Lett.* **B236**, 144 (1990); M. Douglas and S. Shenker, *Nucl. Phys.* **B335**, 635 (1990); L. Alvarez-Gaumé, *Helv. Phys. Acta* **64**, 361 (1991); P. Ginsparg and G. Moore, *Lectures on 2D String Theory and 2D Gravity* (Cambridge, New York, 1993).
4. L. Faddeev and G. Korchemsky, Stony Brook preprint ITP-SB-94-14 (also in hep-th/9404173).
5. S. Manakov, S. Novikov, L. Pitaevski and V. Zakharov, *Soliton Theory: The Inverse Problem* (Nauka, Moscow, 1980).
6. L. A. Dickey, *Soliton Equations and Hamiltonian Systems* (World Scientific, Singapore, 1991).
7. M. Douglas, *Phys. Lett.* **B238**, 176 (1991); E. Martinec, *Commun. Math. Phys.* **138**, 437 (1991); A. Gerasimov, Yu. Makeenko, A. Marshakov, A. Mironov, A. Morozov

[a] Previously, the generalized KP–KdV models were obtained in Ref. 23 from two-matrix models by imposing *ad hoc* additional Dirac constraints on the multiboson KP hierarchy.

and A. Orlov, *Mod. Phys. Lett.* **A6**, 3079 (1991); R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys.* **B348**, 435 (1991).

8. H. Aratyn, E. Nissimov and S. Pacheva, *Phys. Lett.* **B314**, 41 (1993) (also in hep-th/9306035).
9. L. Bonora and C. S. Xiong, *Phys. Lett.* **B285**, 191 (1992) (also in hep-th/9204019).
10. L. Bonora and C. S. Xiong, *Int. J. Mod. Phys.* **A8**, 2973 (1993) (also in hep-th/9209041).
11. L. Bonora and C. S. Xiong, *Nucl. Phys.* **B405**, 191 (1993) (also in hep-th/9212070).
12. L. Bonora and C. S. Xiong, *Phys. Lett.* **B317**, 329 (1993) (also in hep-th/9305005).
13. H. Aratyn, L. A. Ferreira, J. F. Gomes and A. H. Zimerman, *Phys. Lett.* **B316**, 85 (1993) (also in hep-th/9307147).
14. H. Aratyn, L. A. Ferreira and A. H. Zimerman, *Phys. Lett.* **B327**, 266 (1994) (also in hep-th/9312144).
15. M. Toda, *Suppl. Prog. Theor. Phys.* **59**, 1 (1976); *Theory of Nonlinear Lattices* (Springer-Verlag, 1989); M. Wadati, *Suppl. Prog. Theor. Phys.* **59**, 36 (1976); also in *Solitons*, eds. R. K. Bullough and P. J. Caudrey (Springer-Verlag, 1980); S. Kharchev and A. Mironov, *Int. J. Mod. Phys.* **A7**, 4803 (1992).
16. B. A. Kupershmidt, *Commun. Math. Phys.* **99**, 51 (1985).
17. H. Aratyn, L. A. Ferreira, J. F. Gomes, R. T. Medeiros and A. H. Zimerman, *Mod. Phys. Lett.* **A8**, 3079 (1993) (also in hep-th/9302125).
18. L. Bonora and C. S. Xiong, *J. Math. Phys.* **35**, 5781 (1994) (also in hep-th/9311070).
19. M. A. Semenov-Tian-Shansky, *Functional Analysis and Its Application* **17**, 259 (1983).
20. H. Aratyn, E. Nissimov and S. Pacheva, *Phys. Lett.* **B331**, 82 (1984) (also in hep-th/9401058).
21. F. Yu, *Lett. Math. Phys.* **29**, 175 (1993) (also in hep-th/9301053).
22. V. A. Fateev and S. L. Lukyanov, *Int. J. Mod. Phys.* **A3**, 507 (1988).
23. L. Bonora and C. S. Xiong, "Correlation functions of two-matrix models," in hep-th/9311089.
24. H. Aratyn, E. Nissimov, S. Pacheva and A. H. Zimerman, *Phys. Lett.* **B341**, 19 (1994) (also in hep-th/9407017).
25. A. Zamolodchikov, *Theor. Mat. Phys.* **65**, 1205 (1986); V. Fateev and A. Zamolodchikov, *Nucl. Phys.* **B280**, 644 (1987).
26. K. Ueno and K. Takasaki, *Adv. Stud. Pure Math.* **4**, 1 (1984).
27. V. Drinfeld and V. Sokolov, *J. Soviet Math.* **30**, 1975 (1984).
28. J. Balog, L. Fehér, P. Forgács, L. O'Raifeartaigh and A. Wipf, *Ann. Phys.* **203**, 194 (1990); L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, *Phys. Rep.* **222**, 1 (1992).
29. H. Aratyn, L. A. Ferreira, J. F. Gomes and A. H. Zimerman, *Nucl. Phys.* **B402**, 85 (1993) (also in hep-th/9206096); see also *Phys. Lett.* **B293**, 67 (1992) (also in hep-th/9201024).
30. H. Aratyn, L. A. Ferreira, J. F. Gomes and A. H. Zimerman, in *VII J. A. Swieca Summer School — Particles and Fields* (São Paulo, Brazil, 10–23 January 1993), eds. O. J. P. Éboli and V. O. Rivelles (World Scientific, 1994), pp. 630–666 (aso in hep-th/9304152).
31. Y.-S. Wu and F. Yu, *Phys. Rev. Lett.* **68**, 2996 (1992) (also in hep-th/9112009); in *Proc. XXI Int. Conf. on Diff. Geom. Methods in Theor. Phys.* (Nankai Inst. Math., 1992) (also in hep-th/9210162); hep-th/9210117.
32. P. Di Francesco, C. Itzykson and J.-B. Zuber, *Commun. Math. Phys.* **140**, 543 (1991); A. Das and W. J. Huang, *J. Math. Phys.* **33**, 2487 (1992).
33. B. A. Kupershmidt and G. Wilson, *Invent. Math.* **62**, 403 (1981).

34. I. Bakas and E. Kiritsis, *Int. J. Mod. Phys.* **A7** (Suppl. 1A), 55 (1992); F. Yu and Y.-S. Wu, *Nucl. Phys.* **B373**, 713 (1992); *Phys. Lett.* **B294**, 117 (1992).
35. A. G. Reyman, *J. Sov. Math.* **19**, 1507 (1982).
36. D. Bessis, C. Itzykson and J.-B. Zuber, *Adv. Appl. Math.* **1**, 109 (1980); C. Itzykson and J.-B. Zuber, *J. Math. Phys.* **21**, 411 (1980); M. Mehta, *Commun. Math. Phys.* **79**, 327 (1981); S. Chadha, G. Mahoux and M. Mehta, *J. Phys.* **A14**, 579 (1981).
37. B. Konopelchenko, *Phys. Lett.* **A92**, 323 (1982).
38. H. Aratyn, E. Nissimov and S. Pacheva, UICHEP-TH/94-12 (also in hep-th/9501018).